CHAPTER 2

Exponents

CONTENTS

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1 INTRODUCTION

Exponential notation is a short way of writing the same number multiplied by itself many times. This is very useful in everyday life. You may have heard someone describe the size of an area in *square metres* or *square kilometres*. For example, the largest radio telescope in the world is being built in South Africa. The telescope is called the *square kilometre* array, or SKA. This is because the telescope will occupy an area of 1 kilometre by 1 kilometre or 1 kilometre squared.

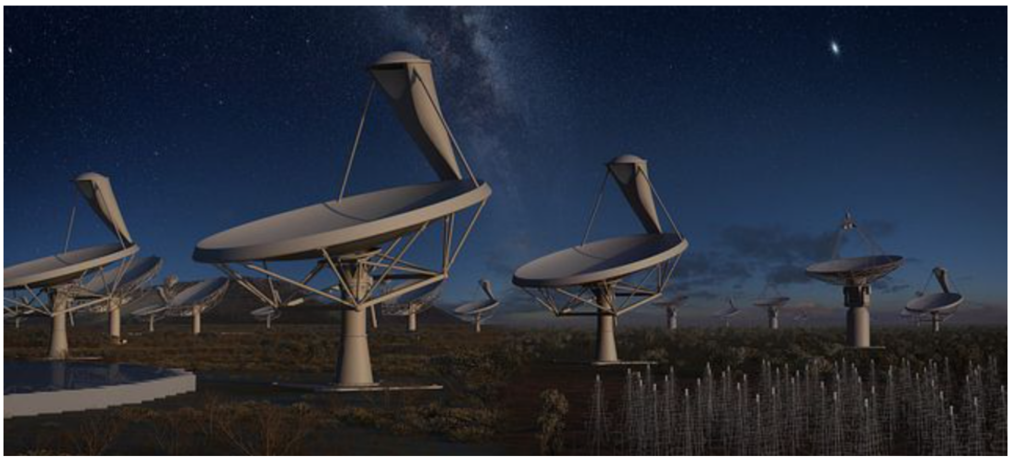

Figure 1: Antennas from the Square Kilometre Array (artist's concept).

Exponents are also very useful to describe very large and very small numbers. For example, the SKA will be detecting incredibly weak signals from objects which are so far away that to write out the strength of the signal or the number of kilometres away in full would be impractical. Outside of astronomy, exponents are used by many other professions such as computer programmers, engineers, economists, financial analysts, biologists and demographers.

You have already been introduced to exponents and exponent laws in previous grades. Remember that exponents can also be called indices or powers. Exponential notation is as follows:

base ← a^n → exponent or index

For any real number a and natural number n, we can write a multiplied by itself n times as: a^n.

Remember the following identities:

1. $a^n = a \times a \times a \times \cdots \times a$ (n times) $(a \in R, n \in N)$
2. $a^0 = 1$ ($a \neq 0$ because 0^0 is undefined)
3. $a^{-n} = \frac{1}{a^n}$ ($a \neq 0$ because $\frac{1}{0}$ is undefined)

4. Similarly, $\frac{1}{a^{-n}} = a^n$

Look at the following examples to see these identities in action:

1. $3 \times 3 = 3^2 = 9$
2. $5 \times 5 \times 5 \times 5 = 5^4$
3. $p \times p \times p = p^3$
4. $(3^x)^0 = 1$
5. $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$
6. $\frac{1}{5^{-x}} = 5^x$

NOTE

If your final answer is easier to work out without a calculator, then write it out in full - not in exponential notation, as in examples 1 and 5.

NOTE

It is convention to write your final answer with positive exponents.

In this chapter, we will revise the exponent laws and use these laws to simplify and solve more complex expressions and equations.

2 REVISION OF EXPONENT LAWS

There are several laws we can use to make working with exponential numbers easier. Some of these laws might have been done in earlier grades, but we list all the laws here for easy reference:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(ab)^n = a^n b^n$
- $(\frac{a}{b})^n = \frac{a^n}{b^n}$
- $(a^m)^n = a^{mn}$

where $a > 0$, $b > 0$ and $m, n \in R$

WORKED EXAMPLE 1: APPLYING THE EXPONENTIAL LAWS

QUESTIONS

Simplify:

1. $2^{3x} \times 2^{4x}$
2. $\frac{4x^3}{2x^5}$
3. $\frac{12p^2t^5}{3pt^3}$
4. $(3x)^2$
5. $(3^4 5^2)^3$
6. $6p^0 \times (7p)^0$
7. $(\frac{2xp}{6x2})^3$
8. $(2^{-2})^{2x+1}$

SOLUTION

1. $2^{3x} \times 2^{4x} = 2^{3x+4x} = 2^{7x}$
2. $\frac{4x^3}{2x^5} = 2x^{3-5} = 2x^{-2} = \frac{2}{x^2}$
3. $\frac{12p^2t^5}{3pt^3} = 4p^{(2-1)}t^{(5-3)} = 4pt^2$
4. $(3x)^2 = 3^2x^2 = 9x^2$
5. $(3^4 5^2)^3 = 3^{(4\times3)} \times 5^{(2\times3)} = 3^{12} \times 5^6$
6. $6p^0 \times (7p)^0 = 6(1) \times 1 = 6$
7. $(\frac{2xp}{6x^2})^3 = (\frac{p}{3x})^3 = \frac{p^3}{27x^3}$
8. $(2^{-2})^{2x+1} = 2^{-2(2x+1)} = 2^{-4x-2}$

WORKED EXAMPLE 2: EXPONENTIAL EXPRESSIONS

QUESTION

Simplify:

$$\frac{2^{2n} \times 4^{n} \times 2}{16^{n}}$$

SOLUTION

Step 1:Change the bases to prime numbers

At first glance it appears that we cannot simplify this expression. However, if we reduce the bases to prime bases, then we can apply the exponent laws.

$$\frac{2^{2n} \times 4^{n} \times 2}{16^{n}} = \frac{2^{2n} \times (2^{2})^{n} \times 2^{1}}{(2^{4})^{n}}$$

Step 2: Simplify the exponents

$$= \frac{2^{2n} \times 2^{2n} \times 2^{1}}{2^{4n}}$$

$$= \frac{2^{2n+2n+1}}{2^{4n}}$$

$$= \frac{2^{4n+1}}{2^{4n}}$$

$$= 2^{4n+1-(4n)}$$

$$= 2$$

TIP

When you have a fraction that is one term over one term, use the method of Finding Prime Bases - in other words use prime factorisation on the bases.

WORKED EXAMPLE 3: EXPONENTIAL EXPRESSIONS

QUESTION

Simplify:

$$\frac{5^{2x-1} \cdot 9^{x-1}}{15^{2x-3}}$$

SOLUTION

Step 1:Change the bases to prime numbers

$$\frac{5^{2x-1} \cdot 9^{x-1}}{15^{2x-3}} = \frac{5^{2x-1} \cdot (3^2)^{x-2}}{(5\times3)^{2x-3}}$$

$$= \frac{5^{2x-1} \cdot 3^{2x-4}}{5^{2x-3} \cdot 3^{2x-3}}$$

Step 2:Subtract the exponents (same base)

$$= 5^{(2x-1)-(2x-3)} \times 3^{(2x-4)-(2x-3)}$$

$$= 5^{2x-1-2x+3} \times 3^{2x-4-2x+3}$$

$$= 5^2 \times 3^{-1}$$

Step 3: Write the answer as a fraction

$$= \frac{5^2}{3}$$

$$= \frac{25}{3}$$

NOTE

When working with exponents, all the laws of operation for algebra apply.

WORKED EXAMPLE 4: SIMPLIFYING BY TAKING OUT A COMMON FACTOR

QUESTION

Simplify:

$$\frac{2^t - 2^{t-2}}{3 \cdot 2^t - 2^t}$$

SOLUTION

Step 1: Simplify to a form that can be factorised

For each of the exponent laws we can "undo" the law - in other words we can work backwards. For this expression we can reverse the multiplication law to write 2^{t-2} as $2^t \cdot 2^{-2}$.

$$\frac{2^t - 2^{t-2}}{3 \cdot 2^t - 2^t} = \frac{2^t - (2^t \cdot 2^{-2})}{3 \cdot 2^t - 2^t}$$

Step 2: Take out a common factor

$$= \frac{2^t(1 - 2^{-2})}{2^t(3 - 1)}$$

Step 3: Cancel the common factor and simplify

$$= \frac{1 - 2^{-2}}{3 - 1}$$

$$= \frac{1 - \frac{1}{4}}{2}$$

$$= \frac{3}{8}$$

When you have a fraction that has more than one term in the numerator or denominator, change to prime bases if necessary and then factorise.

WORKED EXAMPLE 5: SIMPLIFYING USING DIFFERENCE OF TWO SQUARES

QUESTIONS

Simplify:

$$\frac{9^x - 1}{3^x + 1}$$

SOLUTION

Step 1: Change the bases to prime numbers

$$\frac{9^x - 1}{3^x + 1} = \frac{(3^2)^x - 1}{3^x + 1}$$
$$= \frac{3^{2x} - 1}{3^x + 1}$$

Recognise that $3^{2x} = (3^x)^2$

Step 2: Factorise using the difference of squares

$$= \frac{(3^x - 1)(3^x + 1)}{3^x + 1}$$

Step 3: Cancel the common factor and simplify

$$= 3^x - 1$$

3 RATIONAL EXPONENTS

We can also apply the exponent laws to expressions with rational exponents.

WORKED EXAMPLE 6: SIMPLIFYING RATIONAL EXPONENTS

QUESTION

Simplify:

$$2x^{\frac{1}{2}} \times 4x^{\frac{-1}{2}}$$

SOLUTION

$$2x^{\frac{1}{2}} \times 4x^{\frac{-1}{2}} = 8x^{\frac{1}{2} - \frac{1}{2}}$$
$$= 8x^0$$
$$= 8(1)$$
$$= 8$$

WORKED EXAMPLE 7: SIMPLIFYING RATIONAL EXPONENTS

QUESTION

Simplify:

$$(0,008)^{\frac{1}{3}}$$

SOLUTION

Step 1:Write as a fraction and simplify

$$(0,008)^{\frac{1}{3}} = \left(\frac{8}{1000}\right)^{\frac{1}{3}}$$

$$= \left(\frac{1}{125}\right)^{\frac{1}{3}}$$

$$= \left(\frac{1}{5^3}\right)^{\frac{1}{3}}$$

$$= \frac{1^{\frac{1}{3}}}{5^{(3\cdot\frac{1}{3})}}$$

$$= \frac{1}{5}$$

4 EXPONENTIAL EQUATION

Exponential equations have the unknown variable in the exponent. Here are some examples:

$$3^{x+1} = 9$$

$$5^t + 3 \times 5^{t-1} = 400$$

If we can write a single term with the same base on each side of the equation, we can equate the exponents. This is one method to solve exponential equations.

Important: if $a > 0$ and $a \neq 1$ then:

$$a^x = a^y$$

then $x = y$ (same base)

Also notice that if $a = 1$, then x and y can be different.

WORKED EXAMPLE 8: EQUATING EXPONENTS

QUESTION

Solve for x: $3^{x+1} = 9$.

SOLUTION

Step 1:Change the bases to prime numbers

$$3^{x+1} = 3^2$$

Step 2: The bases are the same so we can equate exponents

$$x + 1 = 2$$

$$\therefore x = 1$$

WORKED EXAMPLE 9: EQUATING EXPONENTS

QUESTION

Solve for t: $3^t = 1$.

SOLUTION

Step 1: Solve for t

We know from the exponent identities that $a^0 = 1$, therefore:

$$3^t = 1$$

$$3^t = 3^0$$

$$\therefore t = 0$$

WORKED EXAMPLE 10: SOLVING EQUATIONS BY TAKING OUT A COMMON FACTOR

QUESTION

Solve for t: $5t + 3 \cdot 5t = 400$.

SOLUTION

STEP 1: Rewrite the expression

$$5^t + 3(5^t \cdot 5) = 400$$

Step 2: Take out a common factor

$$5^t(1 + 3 \cdot 5) = 400$$

$$5^t(1 + 15) = 400$$

Step 3: Simplify

$$5^t(16) = 400$$

$$5^t = 25$$

Step 4: Change the bases to prime numbers

$$5^t = 5^2$$

Step 5: The bases are the same so we can equate exponents

$$\therefore t = 2$$

WORKED EXAMPLE 11: SOLVING EQUATIONS BY FACTORISING A TRINOMIAL

QUESTION

Solve for x:

$$3^{2x} - 80 \cdot 3^{x} - 81 = 0$$

SOLUTION

Step 1: Factorise the trinomial

$$(3^{x} - 81)(3^{x} + 1) = 0$$

Step 2: Solve for x

$3^{x} = 81$ or $3^{x} = -1$. However $3^{x} = -1$ is undefined, so:

$$3^{x} = 81$$

$$3^{x} = 3^{4}$$

$$x = 4$$

Therefore $x = 4$

WORKED EXAMPLE 12: SOLVING EQUATIONS BY FACTORISING A TRINOMIAL

QUESTIONS

Solve for p:

$$p - 13p^{\frac{1}{2}} + 36 = 0$$

SOLUTION

Step 1: Rewrite the equation

We notice that $(p^{\frac{1}{2}})^2 = p$ so we can rewrite the equation as:

$$(p^{\frac{1}{2}})^2 - 13p^{\frac{1}{2}} + 36 = 0$$

Step 2: Factorise as a trinomial

$$(p^{\frac{1}{2}} - 9)(p^{\frac{1}{2}} - 4) = 0$$

Step 3: Solve to find both roots

$p^{\frac{1}{2}} - 9 = 0$ or $p^{\frac{1}{2}} - 4 = 0$

$p^{\frac{1}{2}} = 9$ or $p^{\frac{1}{2}} = 4$

$(p^{\frac{1}{2}})^2 = (9)^2$ or $(p^{\frac{1}{2}})^2 = (4)^2$

$p = 81$ or $p = 16$

Therefore $p = 81$ or $p = 16$

WORKED EXAMPLE 13: SOLVING EQUATIONS BY FACTORISATION

QUESTION

Solve for x:

$$2^x - 2^{(4-x)} = 0$$

Step 1 :Rewrite the equation

In order to get the equation into a form which we can factorise, we need to rewrite the equation:

$$2^x - 2^{(4-x)} = 0$$

$$2^x - 2^4 \cdot 2^{-x} = 0$$

$$2^x - \frac{2^4}{2^x} = 0$$

Now eliminate the fraction by multiplying both sides of the equation by the denominator, 2^x.

$$\left(2^x - \frac{2^4}{2^x}\right) \times 2^x = 0 \times 2^x$$

$$2^{2x} - 16 = 0$$

Step 2: Factorise the equation

Now that we have rearranged the equation, we can see that we are left with a difference of two squares. Therefore:

$$2^{2x} - 16 = 0$$

$$(2^x - 4)(2^x + 4) = 0$$

$$2^x = 4$$

or $2^x \neq -4$ (a positive integer with an exponent is always positive)

$$2^x = 2^2$$

$$x = 2$$

Therefore $x = 2$.

5 CHAPTER SUMMARY

- Exponential notation means writing a number as a^n where n is any natural number and a is any real number.
- a is the base and n is the exponent or index.
- Definition:
  - $a^n = a \times a \times \cdots \times a$ (n times)
  - $a^0 = 1$, if $a \neq 0$
  - $a^{-n} = \frac{1}{a^n}$, if $a \neq 0$
  - $\frac{1}{a^{-n}} = a^n$, if $a \neq 0$
- The laws of exponents:
  - $a^m \times a^n = a^{m+n}$
  - $\frac{a^m}{a^n} = a^{m-n}$
  - $(ab)^n = a^n b^n$
  - $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
  - $(a^m)^n = a^{mn}$
- When simplifying expressions with exponents, we can reduce the bases to prime bases or factorise.
- When solving equations with exponents, we can apply the rule that if $a^x = a^y$ then $x = y$; or we can factorise the expressions.

6 EXERCISES

6.1 Exercise 1

1. Simplify without using a calculator:

   1.1 16^0

   1.2 $\dfrac{a^2}{a^{-1}}$

   1.3 $\frac{xy^{-3}}{x^4y}$

   1.4 x^2x^{3t+1}

   1.5 $3 \times 3^{2a} \times 3^2$

   1.6 $\frac{2^{m+20}}{2^{m+20}}$

   1.7 $\frac{2^{x+4}}{2^{x+3}}$

   1.8 $(2a^4)(3ab^2)$

   1.9 $(7m^4n)(8m^6n^8)$

   1.10 $2(-a^7b^8)(-4a^3b^6)(-9a^6b^2)$

   1.11 $(-9x^3y^6)\left(\dfrac{1}{9}x^8y^7\right)\left(\dfrac{1}{5}x^3y^6\right)$

   1.12 $16a^0$

   1.13 $\frac{a^{3x}}{a^x}$

   1.14 $\frac{20x^{10}a^4}{4x^9a^3}$

   1.15 $\frac{18c^{10}p^8}{9c^6p^5}$

   1.16 $\frac{6m^8a^{10}}{2m^3a^5}$

   1.17 $3^{12} \div 3^9$

   1.18 $\dfrac{7(a^3)^3}{a^7}$

   1.19 $\dfrac{9(ab^4)^8}{a^3b^5}$

   1.20 $\frac{2^2}{6^2}$

   1.21 $(\frac{a^6}{b^7})^5$

   1.22 $(2t^4)^3$

   1.23 $11^{9x} \times 11^{2x}$

   1.24 $(3^{n+3})^2$

   1.25 $\dfrac{3^n9^{n-3}}{27^{n-1}}$

1.26 $\frac{13^{c}+13^{c+2}}{3\times13^{c}-13^{c}}$

1.27 $\frac{3^{5x}\times81^{5x}\times3^{3}}{9^{8x}}$

1.28 $\frac{16^{x}-144^{b}}{4^{x}-12^{b}}$

1.29 $\frac{5^{2y-3}2^{4y+4}}{10^{-5y+5}}$

1.30 $\frac{6^{4}\times12^{3}\times4^{5}}{30^{3}\times3^{6}}$

1.31 $\frac{9^{3}\times20^{2}}{4\times5^{2}\times3^{5}}$

1.32 $\frac{7^{b}+7^{b-2}}{4\times7^{b}-3\times7^{b}}$

1.33 $\frac{12^{y}-96^{y}}{3^{y}+6^{y}}$

1.34 $10^{6x}\times10^{2x}$

1.35 $(6c)^{3}$

1.36 $(5n)^{3}$

1.37 $\frac{2^{-2}}{3^{2}}$

1.38 $\frac{5}{2^{-3}}$

1.39 $(\frac{2}{3})^{-3}$

6.2 Exercise 2

1. Simplify without using a calculator:

1. 1 $t^{\frac{1}{4}}\times3t^{\frac{7}{4}}$

1. 2 $\dfrac{16x^{2}}{(4x^{2})^{\frac{1}{2}}}$

1. 3 $(0,25)^{\frac{1}{2}}$

1. 4 $(27)^{-\frac{1}{3}}$

1. 5 $\left(3p^{2}\right)^{\frac{1}{2}}\times\left(3p^{4}\right)^{\frac{1}{2}}$

1. 6 $12\left(a^{4}b^{8}\right)^{\frac{1}{2}}\times\left(512a^{3}b^{3}\right)^{\frac{1}{3}}$

1. 7 $\left((-2)^{4}a^{6}b^{2}\right)^{\frac{1}{2}}$

1. 8 $\left(a^{-2}b^{6}\right)^{\frac{1}{2}}$

1. 9 $\left(16x^{12}b^{6}\right)^{\frac{1}{3}}$

6.3 Exercise 3

1. Solve for the variable:

   1.1 $2^{x+5} = 32$

   1.2 $2^{t} + 2^{t+2} = 40$

   1.3 $(7^{x} - 49)(3^{x} - 27) = 0$

   1.4 $(2 \cdot 2^{x} - 16)\left(3^{x+1} - 9\right) = 0$

   1.5 $(10^{x} - 1)(3^{x} - 81) = 0$

   1.6 $2 \times 5^{2-x} = 5 + 5^{x}$

   1.7 $9^{m} + 3^{3-2m} = 28$

   1.8 $y - 2y^{\frac{1}{2}} + 1 = 0$

   1.9 $4^{x+3} = 0,5$

   1.10 $2^{a} = 0,125$

   1.11 $10^{x} = 0,001$

   1.12 $5^{2x+2} = \frac{1}{125}$

   1.13 $2^{x^{2}-2x-3} = 1$

   1.14 $\frac{8^{x}-1}{2^{x}-1} = 8.2^{x} + 9$

   1.15 $\frac{27^{x}-1}{9^{x}+3^{x}+1} = -\frac{8}{9}$

   1.16 $64^{y+1} = 16^{2y+5}$

   1.17 $3^{9x-2} = 27$

   1.18 $25 = 5^{z-4}$

   1.19 $-\frac{1}{2} \cdot 6^{\frac{m}{2}+3} = -18$

   1.20 $81^{k+2} = 27^{k+4}$

   1.21 $25^{1-2x} - 5^{4} = 0$

   1.22 $27^{x} \times 9^{x-2} = 1$

2. The growth of algae can be modelled by the function $f(t) = 2^{t}$. Find the value of t such that $f(t) = 128$.

3. Use trial and error to find the value of x correct to 2 decimal places.
   $2^{x} = 7$

4. Use trial and error to find the value of x correct to 2 decimal places.
   $5^{x} = 11$

7 ANSWERS FOR EXERCISES

7.1 Exercise 1

1. 1
2. a^3
3. $\frac{1}{x^3y^4}$
4. x^{3t+3}
5. 3^{2a+3}
6. 1
7. 2
8. $6a^5b^2$
9. $56m^10n^9$
10. $-72a^{16}b^{16}$
11. $-\frac{1}{5}x^{14}y^{19}$
12. 16
13. a^{2x}
14. $5ax$
15. $2c^4p^3$
16. $3a^5m^5$
17. 27
18. $7a^2$
19. $9a^5b^{27}$
20. $\frac{1}{9}$
21. $\frac{a^{30}}{b^{35}}$
22. $8t^{12}$
23. 11^{11x}

24. 3^{2n+6}

25. $\frac{1}{27}$

26. 85

27. 3^{9x+3}

28. $4^{x}+12^{b}$

29. $5^{7y-8}\times 2^{9y-1}$

30. $\frac{2^{17}}{3^{2}5^{3}}$

31. 12

32. $\frac{50}{49}$

33. $\frac{4^{y}-2^{5y}}{3}$

34. 10^{8x}

35. $216c^{3}$

36. $125n^{3}$

37. $\frac{1}{36}$

38. 40

39. $\frac{27}{8}$

7.2 Exercise 2

1. $3t^{2}$
2. $8x$
3. $\frac{1}{2}$
4. $\frac{1}{3}$
5. $3p^{3}$
6. $96a^{3}b^{5}$
7. $4a^{3}b$
8. $\frac{b^{3}}{a}$
9. $2\times 2^{\frac{1}{3}}x^{4}b^{2}$

7.3 Exercise 3

1.1 0

1.2 $t = 3$

1.3 $x = 2$ or $x = 3$

1.4 $x = 3$ or $x = 1$

1.5 $x = 0$ or $x = 4$

1.6 $x = 1$ or x = undefined

1.7 $m = \frac{3}{2}$ or $= 0$

1.8 $y = 1$

1.9 $x = -\frac{7}{2}$

1.10 $a = -3$

1.11 $x = -3$

1.12 $x = -\frac{5}{2}$

1.13 $x = 3$ or $x = -1$

1.14 $x = 3$

1.15 $x = -2$

1.16 $y = -7$

1.17 $x = \frac{5}{9}$

1.18 $z = 6$

1.19 $m = -2$

1.20 $k = 4$

1.21 $x = -\frac{1}{2}$

1.22 $x = \frac{4}{5}$

2. $t = 7$

3. $x \approx 2,81$

4. $x \approx 1,49$