CHAPTER 3

Number Patterns

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1 INTRODUCTION

In earlier grades you saw patterns in the form of pictures and numbers. In this chapter, we learn more about the mathematics of patterns. Patterns are repetitive sequences and can be found in nature, shapes, events, sets of numbers and almost everywhere you care to look. For example, seeds in a sunflower, snowflakes, geometric designs on quilts or tiles, or the number sequence $0\ ;\ 4\ ;\ 8\ ;\ 12;\ 16\ ;\ldots$

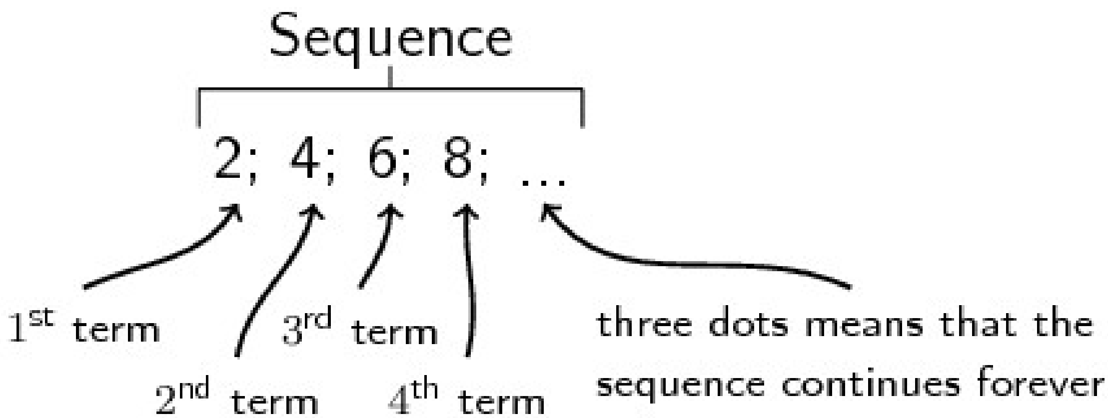
Figure 1: The pattern of seeds within a sunflower follows the Fibonacci sequence, or **1;2;3;5;8;13;21;34;55;89;144;...**

Try to spot any patterns in the following sequences on your own:

1. $2\ ;\ 4\ ;\ 6\ ;\ 8\ ;\ 10\ ;\ldots$
2. $1\ ;\ 2\ ;\ 4\ ;\ 7\ ;\ 11\ ;\ldots$
3. $1\ ;\ 4\ ;\ 9\ ;\ 16\ ;\ 25\ ;\ldots$
4. $5\ ;\ 10\ ;\ 20\ ;\ 40\ ;\ 80\ ;\ldots$

2 DESCRIBING SEQUENCES

A sequence is an ordered list of items, usually numbers. Each item which makes up a sequence is called a “term”.



Sequences can have interesting patterns. Here we examine some types of patterns and how they are formed. Examples:

1. **1;4;7;10;13;16;19;22;25;...** There is a difference of 3 between successive terms. The pattern is continued by adding 3 to the previous term.

2. **13;8;3;−2;−7;−12;−17;−22;...** There is a difference of −5 between successive terms. The pattern is continued by adding −5 to (i.e. subtracting 5 from) the previous term.

3. **2;4;8;16;32;64;128;256;...** This sequence has a factor of 2 between successive terms. The pattern is continued by multiplying the previous term by 2.

4. **3;−9;27;−81;243;−729;2187;...** This sequence has a factor of −3 between successive terms. The pattern is continued by multiplying the previous term by −3.

5. $9;3;1;\frac{1}{3};\frac{1}{9};\frac{1}{27}$**;...** This sequence has a factor of 13 between successive terms. The pattern is continued by multiplying the previous term by $\frac{1}{3}$ which is equivalent to dividing the previous term by 3.

WORKED EXAMPLE 1: STUDY TABLE

Question

You and **3** friends decide to study for Maths and are sitting together at a square table. A few minutes later, **2** other friends arrive and would like to sit at your table. You move another table next to yours so that **6** people can sit at the table. Another **2** friends also want to join your group, so you take a third table and add it to the existing tables. Now **8** people can sit together.

Examine how the number of people sitting is related to the number of tables. Is there a pattern?

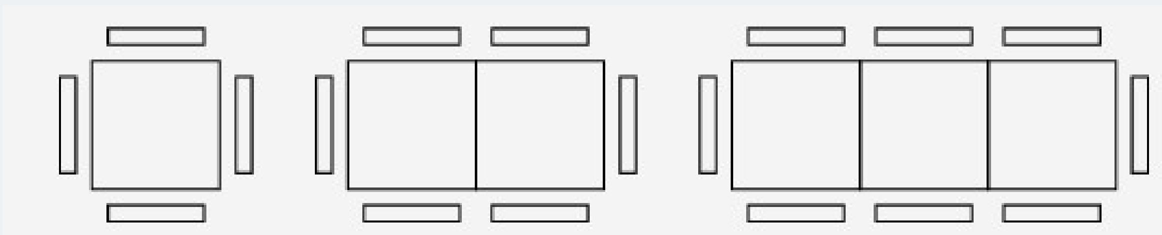


Figure 3.1: Two more people can be seated for each table added.

Solution

Step 1: Make a table to see if a pattern forms

Number of tables,	Number of people seated
1	$4 = 4$
2	$4 + 2 = 6$
3	$4 + 2 + 2 = 8$
4	$4 + 2 + 2 + 2 = 10$
$\vdots$	$\vdots$
n	$4 + 2 + 2 + 2 + \cdots + 2$

Step 2: Describe the pattern

We can see that for **3** tables we can seat **8** people, for **4** tables we can seat **10** people and so on. We started out with **4** people and added two each time. So for each table added, the number of people increased by **2**.

So the pattern formed is **4;6;8;10;...**.

To describe terms in a number pattern we use the following notation:

The first term of a sequence is T_1.

The fourth term of a sequence is T_4.

The tenth term of a sequence is T_{10}.

The general term is often expressed as the n^{th} term and is written as T_n.

A sequence does not have to follow a pattern but, when it does, we can write down the general formula to calculate any term. For example, consider the following linear sequence:
1;3;5;7;9;...

The n^{th} term is given by the general formula:

$$T_n = 2n-1$$

You can check this by substituting values into the formula:

$$T_1 = 2(1) - 1 = 1$$
$$T_2 = 2(2) - 1 = 3$$
$$T_3 = 2(3) - 1 = 5$$
$$T_4 = 2(4) - 1 = 7$$
$$T_5 = 2(5) - 1 = 9$$

If we find the relationship between the position of a term and its value, we can find a general formula which matches the pattern and find any term in the sequence.

Common difference

Consider the following sequence:

6;1;−4;−9;...

We can see that each term is decreasing by 5 but how would we determine the general formula for the n^{th} term? Let us try to do this with a table.

Term number	T_1	T_2	T_3	T_4	T_n
Term	6	1	-4	-9	T_n
Formula	$6 - 0 \times 5$	$6 - 1 \times 5$	$6 - 2 \times 5$	$6 - 3 \times 5$	$6 - (n - 1) \times 5$

You can see that the difference between the successive terms is always the coefficient of n in the formula. This is called a **common difference**.
Therefore, for sequences with a common difference, the general formula will always be of the form: $T_n = dn + c$ where d is the difference between each term and c is some constant.

Note

Sequences with a common difference are called linear sequences.

Definition:

Common difference The common difference is the difference between any term and the term before it. The common difference is denoted by **d**.

For example, consider the sequence **10;7;4;1;...** To calculate the common difference, we find the difference between any term and the previous term.
Let us find the common difference between the first two terms.

$$\begin{aligned} d &= T_2 - T_1 \\ &= 7 - 10 \\ &= -3 \end{aligned}$$

Let us check another two terms:

$$\begin{aligned} d &= T_4 - T_3 \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

We see that **d** is constant.

In general, $d = T_n - T_{n-1}$

Definition:

$d \neq T_{n-1} - T_n$ for example, $d = T_2 - T_1,$ not $T_1 - T_2$

WORKED EXAMPLE 2: STUDY TABLE

QUESTION

As before, you and **3** friends are studying for Maths and are sitting together at a square table. A few minutes later **2** other friends arrive so you move another table next to yours. Now **6** people can sit at the table. Another **2** friends also join your group, so you take a third table and add it to the existing tables. Now **8** people can sit together as shown below.

1. Find an expression for the number of people seated at **n** tables.
2. Use the general formula to determine how many people can sit around **12** tables.
3. How many tables are needed to seat **20** people?

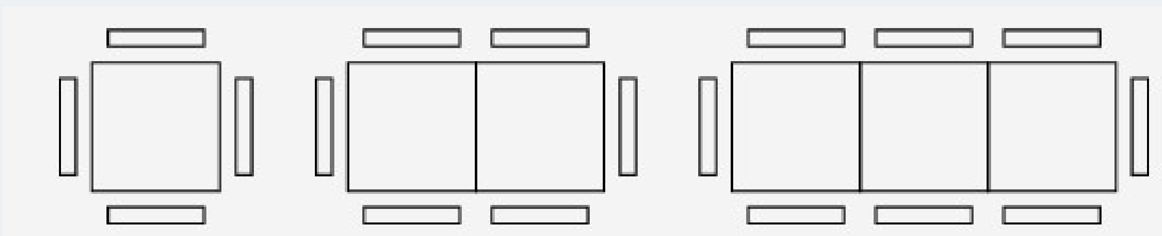


Figure 2: Figure 3.2: Two more people can be seated for each table added.

SOLUTION

Step 1: Make a table to see the pattern

Number of tables,	Number of people seated	Pattern
1	$4 = 4$	$= 4 + 2(0)$
2	$4 + 2 = 6$	$= 4 + 2(1)$
3	$4 + 2 + 2 = 8$	$= 4 + 2(2)$
4	$4 + 2 + 2 + 2 = 10$	$= 4 + 2(3)$
$\vdots$	$\vdots$	$\vdots$
n	$4 + 2 + 2 + 2 + \cdots + 2$	$= 4 + 2(n-1)$

Note: There may be variations in how you think of the pattern in this problem. For example, you may view this problem as the person on one end fixed, two people seated opposite each other per table and one person at the other end fixed. This results in **1+2n+1=2n+2.** Your formula for T_n will still be correct.

Step 2: Describe the pattern : The number of people seated at **n** tables is $T_n = 4 + 2(n-1)$

WORKED EXAMPLE 3: DATA PLANS CONTINUED:

Step 3: Calculate the 12th term, in other words, find T_n if $n = 12$

$$\begin{aligned} T_{12} &= 4 + 2(12 - 1) \\ &= 4 + 2(11) \\ &= 4 + 22 \\ &= 26 \end{aligned} \qquad (1)$$

Therefore 26 people can be seated at 12 tables.

Step 4: Calculate the number of tables needed to seat 20 people, in other words, find n if $T_n = 20$

$$\begin{aligned} T_n &= 4 + 2(n - 1) \\ 20 &= 4 + 2(n - 1) \\ 20 &= 4 + 2n - 2 \\ 20 - 4 + 2 &= 2n \\ 18 &= 2n \\ \frac{18}{2} &= n \\ n &= 9 \end{aligned} \qquad (2)$$

Therefore 9 tables are needed to seat 20 people

It is important to note the difference between n and T_n. n can be compared to a place holder indicating the position of the term in the sequence, while T_n is the value of the place held by n. From our example above, the first table holds 4 people. So for $n = 1$, the value of $T_1 = 4$ and so on:

n	1	2	3	4	...
Tn	4	6	8	10	...

WORKED EXAMPLE 3: DATA PLANS

QUESTION

Raymond subscribes to a limited data plan from Vodacell. The limited data plans cost $R120$ for 1 gigabyte (GB) per month, $R135$ for 2 GB per month and $R150$ for 3 GB per month. Assume this pattern continues indefinitely.

1. Use a table to set up the pattern of the cost of the data plans.
2. Find the general formula for the sequence.
3. Use the general formula to determine the cost for a $30GB$ data plan.
4. The cost of an unlimited data plan is $R520$ per month. Determine the amount of data Raymond would have to use for it to be cheaper for him to sign up for the unlimited plan.

Step 1: Make a table to see the pattern

Number of GB(n)	1	2	3	4
$Cost(inRands)$	120	135	150	165
$Pattern$	120	$120+(1)(15)=135$	$120+(2)(15)=150$	$120+(3)(15)=165$

Step 2: Use the observed pattern to determine the general formula.

The price of n GB of data is $T_n = 120 + 15(n-1)$

Step 3: Determine the cost of 30 GB of data.

This question requires us to determine the value of the 30^{th} term, in other words, find T_n if $n = 30$. Using the general formula, we get:

$$\begin{aligned} T_n &= 120 + 15(n-1) \\ T_{30} &= 120 + 15(30-1) \\ &= 120 + 15(29) \\ &= 120 + 435 \\ &= 555 \end{aligned}$$

Therefore the cost of a **30 GB** data package is **R555.**

WORKED EXAMPLE 3: DATA PLANS CONTINUED

Step 4: Determine when it is cheaper to purchase the unlimited data plan

The final question of this worked example requires us to determine when it would be cheaper for Raymond to purchase an unlimited data plan instead of a limited plan. In other words, we need to find n where T_n is less than **R520**.

We know that:

$$T_n = 120 + 15(n-1)$$

Therefore, if $T_n = 520 = 120 + 15(n-1)$

Solving for n, we get:

$$\begin{aligned} 520 &= 120 + 15(n-1) \\ 520 &= 120 + 15(n-15) \\ 520 &= 105 + 15n \\ 405 &= 15n \\ \frac{405}{15} &= n \\ n &= 27 \end{aligned} \quad (3)$$

Therefore it is cheaper for Raymond to purchase the unlimited data plan if he uses more than $27GB$ per month.

3 CHAPTER SUMMARY

- The general term is expressed as the nth term and is written as T_n.
- We define the common difference d of a sequence as the difference between any two successive terms, where $d = T_n - T_{n-1}$.
- We can work out a general formula for each number pattern and use it to determine any term in the pattern.

4 EXERCISES

4.1 Exercise 1

1. Use the given pattern to complete the table below.

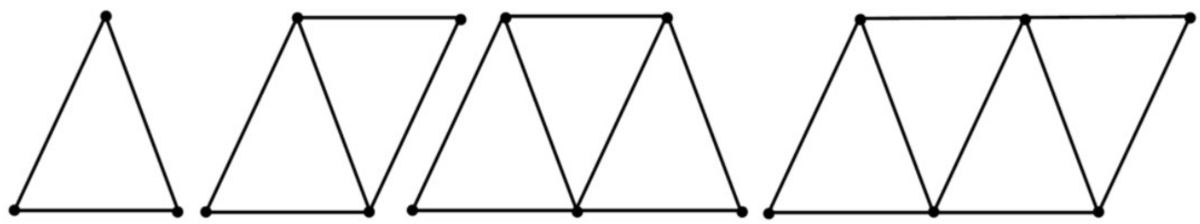


Figure number	1	2	3	4	n
Number of dots					
number of lines					
Total					

2. Consider the sequence shown here:

$$-4\,;\ -1\,;\ 2\,;\ 5\,;\ 8\,;\ 11\,;\ 14\,;\ 17\,;\ \ldots$$

If $T_n = 2$ what is the value of T_{n-1} ?

3. Consider the sequence shown here:

$$C;\ D;\ E;\ F;\ G;\ H;\ I;\ J;\ \ldots$$

If $T_n = G$ what is the value of T_{n-4} ?

4. For each of the following determine the common difference. If the sequence is not linear, write "no common difference".

4.1 $9\,;\ -7\,;\ -8\,;\ -25\,;\ -34\,;\ \ldots$

4.2 $5\,;\ 12\,;\ 19\,;\ 26\,;\ 33\,;\ \ldots$

4.3 $2{,}93\,;\ 1{,}99\,;\ 1{,}14\,;\ 0{,}35\,;\ \ldots$

4.4 $2{,}53\,;\ 1{,}88\,;\ 1{,}23\,;\ 0{,}58\,;\ \ldots$

5. Write down the next three terms in each of the following sequences:

   5.1 $5\ ;\ 15\ ;\ 25\ ;\ \ldots$

   5.2 $-8\ ;\ -3\ ;\ 2\ ;\ \ldots$

   5.3 $30\ ;\ 27\ ;\ 24\ ;\ \ldots$

   5.4 $-13,1\ ;\ -18,1\ ;\ -23,1\ ;\ldots$

   5.5 $-9x\ ;-19x\ ;-29x\ ;\ \ldots$

   5.6 $-15,8\ ;\ 4,2\ ;\ 24,2\ ;\ldots$

   5.7 $30b\ ;34b\ ;38b\ ;\ \ldots$

6. Given a pattern which starts with the numbers:
   $3;\ 8;\ 13;\ 18;\ \ldots$
   determine the values of T_6 and T_9

7. Given a sequence which starts with the letters:
   $C;\ D;\ E;\ F;\ \ldots$
   determine the values of T_5 and T_8

8. Given a pattern which starts with the numbers:
   $7;\ 11;\ 15;\ 19;\ \ldots$
   determine the values of T_5 and T_8

4.2 Exercise 2

1. The general term is given for each sequence below. Calculate the missing terms (each missing term is represented by . . .).

   1.1 $0\ ;\ 3\ ;\ \ldots\ ;\ 15\ ;\ 24 \qquad T_n = n^2 - 1$

   1.2 $3\ ;\ 2\ ;\ 1\ ;\ 0\ ;\ \ldots;-2 \qquad T_n = -n + 4$

   1.3 $-11\ ;\ \ldots\ ;\ -7\ ;\ \ldots\ ;\ -3 \qquad T_n = -13 + 2n$

   1.4 $1\ ;\ 10\ ;\ 19\ ;\ \ldots\ ;\ 37 \qquad T_n = 9n - 8$

   1.5 $9\ ;\ \ldots\ ;\ 21\ ;\ \ldots\ ;\ 33 \qquad T_n = 6n + 3$

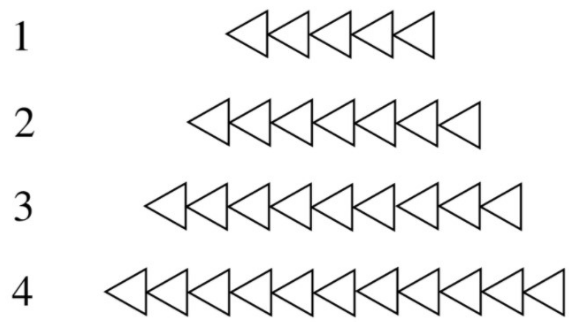
2. Find the general formula for the following sequences and then find T_{10} , T_{50} and T_{100} .

   2.1 $2;\ 5;\ 8;\ 11;\ 14;\ \ldots$

   2.2 $0;\ 4;\ 8;\ 12;\ 16;\ \ldots$

   2.3 $2;\ -1;\ -4;\ -7;\ -10;\ \ldots$

3. The diagram below shows pictures which follow a pattern.



   3.1 How many triangles will there be in the 5^{th} picture?

   3.2 Determine the formula for the n^{th} term.

   3.3 Use the formula to find how many triangles are in the 25^{th} picture of the diagram.

4. Study the following sequence:
   $15\ ;\ 23\ ;\ 31\ ;\ 39\ ;\ \ldots$

   4.1 Write down the next 3 terms.

   4.2 Find the general formula for the sequence.

   4.3 Find the value of n if T_n is 191 .

5. Study the following sequence:
   $-44\ ;\ -14\ ;\ 16\ ;\ 46\ ;\ \ldots$

   5.1 Write down the next 3 terms.

   5.2 Find the general formula for the sequence.

   5.3 Find the value of n if T_n is 406 .

4.3 Exercise 3

1. Consider the following list:
   $-z-5\ ;\ -4z-5\ ;\ -6z-2\ ;\ -8z-5\ ;\ -10z-5\ ;\ \ldots$

   1.1 Find the common difference for the terms of the list. If the sequence is not linear (if it does not have a common difference), write “no common difference”.

   1.2 If you are now told that $z = -2$, determine the values of T_1 and T_2 .

2. Consider the following pattern:

   $2n + 4;\ 1;\ -2n - 2;\ -4n - 5;\ -6n - 8;\ \ldots$

   2.1 Find the common difference for the terms of the pattern. If the sequence is not linear (if it does not have a common difference), write "no common difference".

   2.2 If you are now told that $n = -1$, determine the values of T_1 and T_3 .

3. If the following terms make a linear sequence:

   $\frac{k}{3} - 1\ ;\ -\frac{5k}{3} + 2\ ;\ -\frac{2k}{3} + 10\ ;\ldots$

   3.1 Determine the value of k . If the answer is a non-integer, write the answer as a simplified fraction.

   3.2 Now determine the numeric value of the first three terms. If the answers are not integers, write your answers as fractions.

4. Answer the following questions:

   4.1 If the following terms make a linear sequence: $y - \frac{3}{2}\ ;\ -y - \frac{7}{2}\ ;\ -7y - \frac{15}{2}\ ;\ \ldots$ find y . If the answer is a non-integer, write the answer as a simplified fraction.

   4.2 Now determine the numeric value of the first three terms. If the answers are not integers, write your answers as fractions.

5. What is the 649^{th} letter of the sequence:

   PATTERNPATTERNPATTERNPATTERNPATTERNPATTERNPATTE............?

5 ANSWERS TO EXERCISES

5.1 Exercise 1

1.

Figure number	1	2	3	4	n
number of dots	3	4	5	6	$n+2$
number of lines	3	5	7	9	$2n+1$
Total	6	9	12	15	$3(n+1)$

2. $T_{n-1} = -1$
3. C
4. 4.1 No common difference.
   4.2 $d = 7$
   4.3 No common difference
   4.4 $d = -0,65$
5. 5.1 $5\ ;\ 15\ ;\ 25\ ;\ 35\ ;\ 45\ ;\ 55\ ;\ \ldots$
   5.2 $8\ ;\ -3\ ;\ 2\ ;\ 7\ ;\ 12\ ;\ 17\ ;\ \ldots$
   5.3 $30;\ 27;\ 24;\ 21;\ 18;\ 15;\ \ldots$
   5.4 $-28,1;\ -33,1;\ -38,1$
   5.5 $-39x;\ -49x;\ -59x$
   5.6 $44,2;\ 64,2;\ 84,2$
   5.7 $42b;\ 46b;\ 50b$
6. $T_6 = 28$ and $T_9 = 43$
7. $T_5 = G$ and $T_8 = J$
8. $T_5 = 23$ and $T_8 = 35$

5.2 Exercise 2

1. 1.1 $0\ ;3\ ;\ 8\ ;\ 15\ ;\ 24$
   1.2 -1
   1.3 -9 and -5
   1.4 28

1.5 $T_2 = 15$ and $T_4 = 27$

2. 2.1 $T_{10} = 29$
   $T_{50} = 149$
   $T_{100} = 299$

   2.2 $T_{10} = 36$
   $T_{50} = 196$
   $T_{100} = 396$

   2.3 $T_{10} = -25$
   $T_{50} = -145$
   $T_{100} = -295$

3. 3.1 13 triangles

   3.2 $T_n = 2n + 3$

   3.3 $T_{25} = 53$

4. 4.1 $47;\ 55;\ 63$

   4.2 $T_n = 8n + 7$

   4.3 $n = 23$

5. 5.1 $76\ ;\ 106\ ;\ 136$

   5.2 $30n - 74$

   5.3 $n = 16$

5.3 Exercise 3

1. 1.1 No common difference.

   1.2 $T_1 = -3$
   $T_2 = 3$

2. 2.1 $d = -2n - 3$

   2.2 $T_1 = 2$
   $T_3 = 0$

3. 3.1 $k = -\frac{5}{3}$

   3.2 $-\frac{14}{9}, \frac{43}{9}$ and $\frac{100}{9}$

4. 4.1 $y = -\frac{1}{2}$

   4.2 $-2, -3$ and -4

5. 5^{th} letter which is E .