# CHAPTER 5

*Trigonometry*

MathU Teaching Emporium®

# CONTENTS

# 1 INTRODUCTION

Trigonometry deals with the relationship between the angles and sides of a triangle. We will learn about trigonometric ratios in right-angled triangles, which form the basis of trigonometry.
There are many applications of trigonometry. Of particular value is the technique of triangulation, which is used in astronomy to measure the distances to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. GPS (the global positioning system) would not be possible without trigonometry. Other fields which make use of trigonometry include acoustics, optics, analysis of financial markets, electronics, probability theory, statistics, biology, medical imaging (CAT scans and ultrasound), chemistry, cryptology, meteorology, oceanography, land surveying, architecture, phonetics, engineering, computer graphics and game development.

Figure 1: An artist's depiction of a GPS satellite orbiting the Earth. There are at least 24 GPS satellites operational at any one time. GPS uses an application of trigonometry, known as triangulation, to determine one's position. The accuracy of GPS is to within 15 metres.

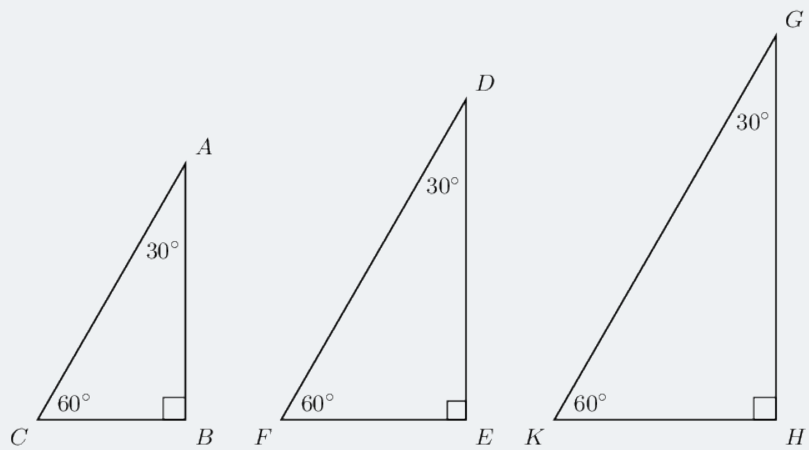
# 2 SIMILARITY OF TRIANGLES

Before we delve into the theory of trigonometry, complete the following investigation to get a better understanding of the foundation of trigonometry.

INVESTIGATION

**Ratios of similar triangles**

Draw three similar triangles of different sizes using a protractor and a ruler, with each triangle having interior angles equal to $30°$, $90°$ and $60°$ as shown below. Measure the angles and lengths accurately in order to fill in the table (leave your answers as a simplified fraction):

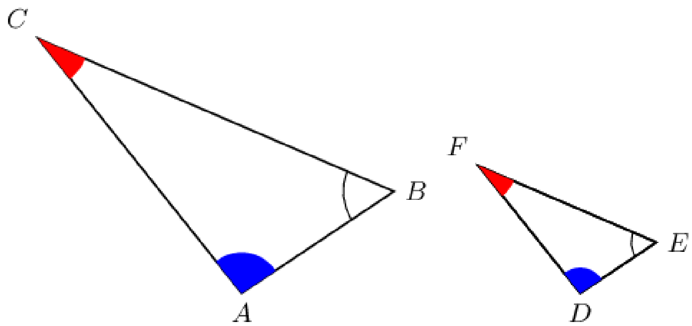



| Dividing lengths of sides (ratios) | | | |
|---|---|---|---|
| | $\frac{AB}{BC} =$ | $\frac{AB}{AC} =$ | $\frac{CB}{AC} =$ |
| | $\frac{DE}{EF} =$ | $\frac{DE}{DF} =$ | $\frac{FE}{DF} =$ |
| | $\frac{GH}{HK} =$ | $\frac{GH}{GK} =$ | $\frac{KH}{GK} =$ |

What observations can you make about the ratios of the sides?

Have you noticed that it does not matter what the lengths of the sides of the triangles are, if the angle remains constant, the ratio of the sides will always yield the same answer?

In the triangles below, $\triangle ABC$ is similar to $\triangle DEF$. This is written as:
$\triangle ABC \ ||| \ \triangle DEF$




In similar triangles, it is possible to deduce ratios between corresponding sides:

$$\frac{AB}{BC} = \frac{DE}{EF}$$
$$\frac{AB}{AC} = \frac{DE}{DF}$$
$$\frac{AC}{BC} = \frac{DF}{EF}$$
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Another important fact about similar triangles $ABC$ and $DEF$ is that the angle at vertex $A$ is equal to the angle at vertex $D$, the angle at vertex $B$ is equal to the angle at vertex $E$, and the angle at vertex $C$ is equal to the angle at vertex $F$.

$$\hat{A} = \hat{D}$$
$$\hat{B} = \hat{E}$$
$$\hat{C} = \hat{F}$$

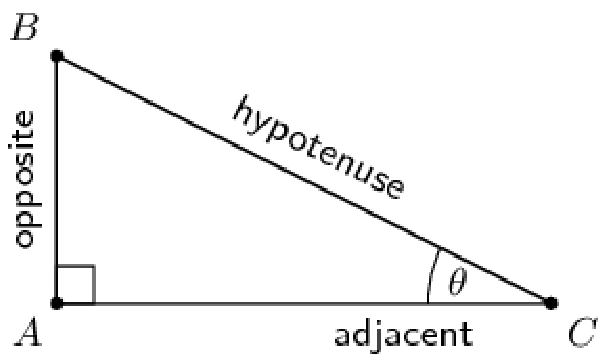
**NOTE**

The order of letters for similar triangles is very important. Always label similar triangles in corresponding order. For example,

$\triangle ABC \ ||| \ \triangle DEF$ is correct; but
$\triangle ABC \ ||| \ \triangle DFE$ is incorrect

# 3 DEFINING THE TRIGONOMETRIC RATIOS

The ratios of similar triangles are used to define the trigonometric ratios. Consider a right-angled triangle $\triangle ABC$ with an angle marked $\theta$ (said 'theta').




In a right-angled triangle, we refer to the three sides according to how they are placed in relation to the angle $\theta$. The side opposite to the right-angle is labelled the hypotenuse, the side opposite $\theta$ is labelled "opposite", the side next to $\theta$ is labelled "adjacent".

You can choose either non-$90°$ internal angle and then define the adjacent and opposite sides accordingly. However, the hypotenuse remains the same regardless of which internal angle you are referring to because it is *always* opposite the right-angle and *always* the longest side.

We define the trigonometric ratios: sine (sin), cosine (cos) and tangent (tan), of an angle, as follows:

$$\sin\theta = \frac{opposite}{hypotenuse}$$
$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$\tan\theta = \frac{opposite}{adjacent}$$

These ratios, also known as trigonometric identities, relate the lengths of the sides of a right-angled triangle to its interior angles. These three ratios form the basis of trigonometry.

**IMPORTANT**

The definitions of opposite, adjacent and hypotenuse are only applicable when working with right-angled triangles! Always check to make sure your triangle has a right-angle before you use them, otherwise you will get the wrong answer.

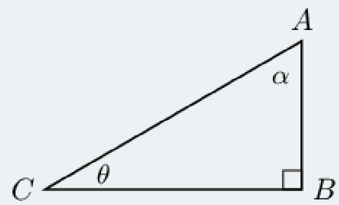
You may also hear people saying "Soh Cah Toa". This is a mnemonic technique for remembering the trigonometric ratios:

| |
|---|
| $\sin\theta = \frac{opposite}{hypotenuse}$ |
| $\cos\theta = \frac{adjacent}{hypotenuse}$ |
| $\tan\theta = \frac{opposite}{adjacent}$ |

WORKED EXAMPLE 1: TRIGNOMETRIC RATIOS

Question

Given the following triangle:



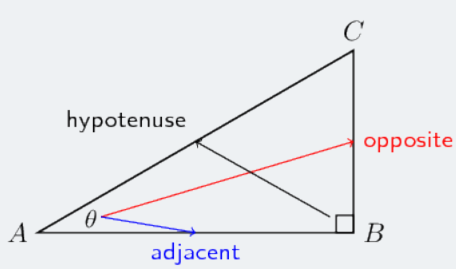

- Label the hypotenuse, opposite and adjacent sides of the triangle with respect to $\theta$.
- State which sides of the triangle you would use to find $\sin\theta$, $\cos\theta$ and $\tan\theta$.
- Label the hypotenuse, opposite and adjacent sides of the triangle with respect to $\alpha$.
- State which sides of the triangle you would use to find $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$.

Solution

**Step 1: Label the triangle**

First find the right angle, the hypotenuse is **always** directly opposite the right angle. The hypotenuse never changes position, it is always directly opposite the right angle and so we find this first.

The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle $\theta$ is directly opposite (as the word opposite suggests) the angle $\theta$. Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle $\theta$.



WORKED EXAMPLE 1: TRIGNOMETRIC RATIOS (CONTINUED)

**Step 2: Complete the trigonometric ratios**

Now we can complete the trigonometric ratios for $\theta$:

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{CB}{AC}$$
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{AB}{AC}$$
$$\tan\theta = \frac{opposite}{adjacent} = \frac{CB}{AB}$$

Therefore to find $\sin\theta$ we would use sides $CB$ (opposite side to $\theta$) and $AC$ (hypotenuse). To find $\cos\theta$ we would use sides $AB$ (adjacent side to $\theta$) and $AC$ (hypotenuse). To find $\tan\theta$ we would use sides $CB$ (opposite side to $\theta$) and $AB$ (adjacent side to $\theta$).

And then we can complete the trigonometric ratios for $\alpha$. For angle $\alpha$ the opposite and adjacent sides switch places (redraw the triangle above to help you see this). Notice how the hypotenuse is still $AC$.

$$\sin\alpha = \frac{opposite}{hypotenuse} = \frac{AB}{AC}$$
$$\cos\alpha = \frac{adjacent}{hypotenuse} = \frac{CB}{AC}$$
$$\tan\alpha = \frac{opposite}{adjacent} = \frac{AB}{CB}$$

Therefore to find $\sin\alpha$ we would use sides $AB$ (opposite side to $\alpha$) and $AC$ (hypotenuse). To find $\cos\alpha$ we would use sides $CB$ (adjacent side to $\alpha$) and $AC$ (hypotenuse). To find $\tan\alpha$ we would use sides $AB$ (opposite side to $\alpha$) and $CB$ (adjacent side to $\alpha$).

# 4 RECIPROCAL RATIOS

Each of the three trigonometric ratios has a reciprocal. The reciprocals: cosecant (csc), secant (sec) and cotangent (cot), are defined as follows:

$$csc\theta = \frac{1}{\sin\theta}$$
$$sec\theta = \frac{1}{\cos\theta}$$
$$cot\theta = \frac{1}{\tan\theta}$$

We can also define these reciprocals for any right-angled triangle:

$$csc\theta = \frac{hypotenuse}{opposite}$$
$$sec\theta = \frac{hypotenuse}{adjacent}$$
$$cot\theta = \frac{adjacent}{opposite}$$

Note that:

$$\sin\theta \times \csc\theta = 1$$
$$\cos\theta \times \sec\theta = 1$$
$$\tan\theta \times \cot\theta = 1$$

**NOTE**

You may see cosecant abbreviated as $csc$.

# 5 CALCULATOR SKILLS

In this section we will look at using a calculator to determine the values of the trigonometric ratios for any angle. For example we might want to know what the value of $\sin 55°$ is or what the value of $\sec 34°$ is.

When doing calculations involving the reciprocal ratios you need to convert the reciprocal ratio to one of the standard trigonometric ratios: sin, cos and tan as this is the only way to calculate these ratios on your calculator.

IMPORTANT

Most scientific calculators are quite similar but these steps might differ depending on the calculator you use. Make sure your calculator is in "degrees" mode.

TIP

Note that $\sin^2\theta = (\sin\theta)^2$. This also applies for the other trigonometric ratios.

WORKED EXAMPLE 2: USING YOUR CALCULATOR

Question

Use your calculator to calculate the following (correct to 2 decimal places):

1. $\cos 48°$
2. $2\sin 35°$
3. $\tan^2 81°$
4. $3\sin^2 72°$
5. $\frac{1}{4}\cos 27°$
6. $\frac{5}{6}\tan 34°$
7. $\sec 34°$
8. $\cot 49°$

Solution

**Step 1:**

The following shows the keys to press on a Casio calculator. Other calculators work in a similar way. On a Casio calculator ( is automatically added after pressing $sin$, $cos$ and $tan$ so you just need to press ) after typing in the angle to close the brackets.

1. Press cos 48 ) = 0.66913... $\approx$ 0.67

2. Press 2 sin 35 ) = 1.147152... $\approx$ 1.15

3. Press ( tan 81 ) ) $x^2$ = 39.86345... $\approx$ 39.86

4. Press 3 ( sin 72 ) ) $x^2$ = 2.71352... $\approx$ 2.71

WORKED EXAMPLE 2: USING YOUR CALCULATOR (CONTINUED)

5. Press ( 1 ÷ 4 ) cos 27 ) = 0.22275... $\approx$ 0.22

6. Press ( 5 ÷ 6 ) tan 34 ) = 0.56209... $\approx$ 0.56

7. First write sec in terms of cos: $\sec 34° = \frac{1}{\cos 34°}$ (since there is no sec button on your calculator).
   Press 1 ÷ ( cos 34 ) ) = 1.206217... $\approx$ 1.21

8. First write cot in terms of tan: $\cot 49° = \frac{1}{\tan 49°}$ (since there is no cot button on your calculator).
   Press 1 ÷ ( tan 49 ) ) = 0.869286... $\approx$ 0.87

WORKED EXAMPLE 3: CALCULATOR WORK USING SUBSTITUTION

Question

If $x = 25°$ and $y = 65°$, use your calculator to determine whether the following statement is true or false:

$$\sin^2 x + \cos^2 (90° - y) = 1$$

Solution

**Step 1: Calculate the left hand side of the equation**

Press [(] [sin] [25] [)] [)] [x²] [+] [(] [cos] [90] [−] [65] [)] [)] [x²] [=] 1

**Step 2: Write the final answer**

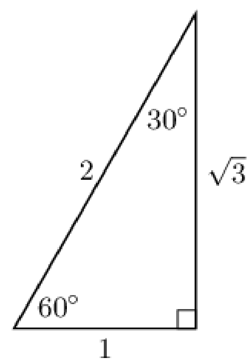
LHS = RHS therefore the statement is true.

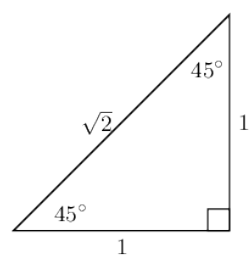
# 6 SPECIAL ANGLES

For most angles we need a calculator to calculate the values of $sin$, $cos$ and $tan$. However, there are some angles we can easily work out the values for without a calculator as they produce simple ratios. The values of the trigonometric ratios for these special angles, as well as the triangles from which they are derived, are shown below.

NOTE

Remember that the lengths of the sides of a right-angled triangle must obey the Theorem of Pythagoras: the square of the hypotenuse equals the sum of the squares of the two other sides.




| $\theta$ | $30°$ | $45°$ | $60°$ |
|---|---|---|---|
| $\cos\theta$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ |
| $\sin\theta$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ |
| $\tan\theta$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ |

These values are useful when we need to solve a problem involving trigonometric ratios without using a calculator.

# 7 SOLVING TRIGONOMETRIC EQUATIONS

In this section we will first look at finding unknown lengths in right-angled triangles and then we will look at finding unknown angles in right-angled triangles. Finally we will look at how to solve more general trigonometric equations.

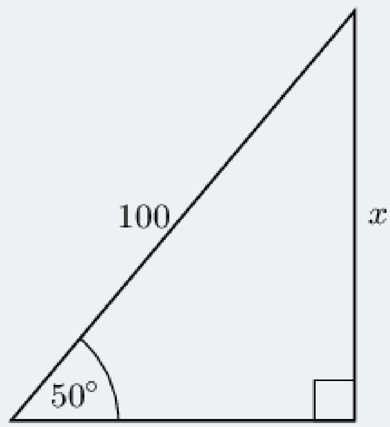
**Finding lengths**

From the definitions of the trigonometric ratios and what we have learnt about determining the values of these ratios for any angle we can now use this to help us find unknown lengths in right-angled triangles. The following worked examples will show you how.

WORKED EXAMPLE 4: FINDING LENGTHS

Question

Find the length of $x$ in the following right-angled triangle using the appropriate trigonometric ratio (round your answer to two decimal places).




Solution **Step 1: Identify the opposite and adjacent sides and the hypotenuse with reference to the given angle**

Remember that the hypotenuse side is **always** opposite the right angle, it never changes position. The opposite side is opposite the angle we are interested in and the adjacent side is the remaining side.

$$\sin\theta = \frac{opposite}{hypotenuse}$$
$$\sin 50^\circ = \frac{x}{100}$$

**Step 2: Rearrange the equation to solve for $x$**

$$\sin 50^\circ \times 100 = \frac{x}{100} \times 100$$
$$\sin 50^\circ \times 100 = x$$
$$x = 100\sin 50^\circ$$

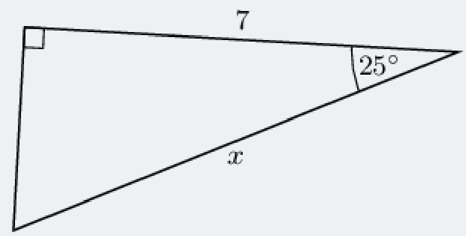
**Step 3: Use your calculator to find the answer**

$$x = 76.60444...$$
$$x \approx 76.60$$

## WORKED EXAMPLE 5: FINDING LENGTHS

Question

Find the length of $x$ in the following right-angled triangle using the appropriate trigonometric ratio (round your answer to two decimal places).




Solution

**Step 1: Identify the opposite and adjacent sides and the hypotenuse with reference to the given angle**

Remember that the hypotenuse side is **always** opposite the right angle, it never changes position. The opposite side is opposite the angle we are interested in and the adjacent side is the remaining side.

$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$\cos 25^\circ = \frac{7}{x}$$

**Step 2: Rearrange the equation to solve for $x$**

$$\cos 25^\circ \times x = \frac{7}{x} \times x \quad \text{multiply both sides by x}$$
$$x \cos 25^\circ = 7$$
$$\frac{x \cos 25^\circ}{\cos 25^\circ} = \frac{7}{\cos 25^\circ} \quad \text{divide both sides by } \cos 25^\circ$$
$$x = \frac{7}{\cos 25^\circ}$$

**Step 3: Use your calculator to find the answer**



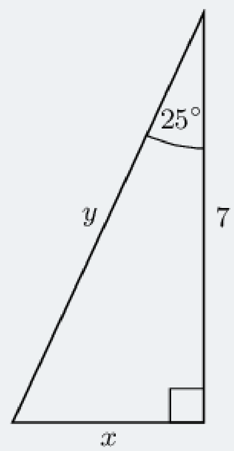


$$x = \frac{7}{0.90630...}$$
$$= 7.723645...$$

WORKED EXAMPLE 6: FINDING LENGTHS

Question

Find the length of $x$ and $y$ in the following right-angled triangle using the appropriate trigonometric ratio (round your answers to two decimal places).




Solution

**Step 1: Identify the opposite and adjacent sides and the hypotenuse with reference to the given angle**

$$\tan\theta = \frac{opposite}{adjacent}$$
$$\tan 25^\circ = \frac{x}{7}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$\cos 25^{circ} = \frac{7}{y}$$

**Step 2: Rearrange the equations to solve for $x$ and $y$**

$$x = 7 \times \tan 25^\circ$$
$$y = \frac{7}{\cos 25^\circ}$$

**Step 3: Use your calculator to find the answers**

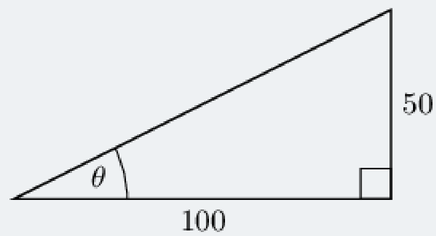
# 8 FINDING AN ANGLE

If the length of two sides of a triangle are known, the angles can be calculated using trigonometric ratios. In this section, we are finding angles inside right-angled triangles using the ratios of the sides.

## WORKED EXAMPLE 7: FINDING ANGLES

Question

Find the value of $\theta$ in the following right-angled triangle using the appropriate trigonometric ratio.




Solution

**Step 1: Identify the opposite and adjacent sides with reference to the given angle and the hypotenuse**

In this case you have the opposite side and the adjacent side for angle $\theta$.

$$\tan\theta = \frac{opposite}{adjacent}$$
$$\tan\theta = \frac{50}{100}$$

**Step 2: Use your calculator to solve for $\theta$**

To solve for $\theta$, you will need to use the inverse tangent function on your calculator. This works backwards by using the ratio of the sides to determine the angle which resulted in that ratio.

Press [Shift] [tan] [50] [$\div$] [100] [)] [=] 26.56505... $\approx$ 26.6

**Step 3: Write the final answer**

$\theta = 26.6°$

We have now seen how to solve trigonometric equations in right-angled triangles. We can use the same techniques to help us solve trigonometric equations when the triangle is not shown.

WORKED EXAMPLE 8: SOLVING TRIGNOMETRIC EQUATIONS

Question

Find the value of $\theta$ if $\cos\theta = 0,2$.

Solution

**Step 1: Use your calculator to solve for $\theta$**

To solve for $\theta$, you will need to use the inverse cosine function on your calculator. This works backwards by using the ratio of the sides to determine the angle which resulted in that ratio.

Press [shift] [cos] [0] [.] [2] [)] [=] 78.46304 ≈ 78.46

**Step 2: Write the final answer**

$\theta = 78.46°$

## WORKED EXAMPLE 9: SOLVING TRIGNOMETRIC EQUATIONS

Question

Find the value of $\theta$ if $3\sin\theta = 2.4$.

Solution

**Step 1: Rearrange the equation**

We need to rearrange the equation so that $sin\theta$ is on one side of the equation.

$$3\sin\theta = 2.4$$
$$\sin\theta = \frac{2.4}{3}$$

**Step 2: Use your calculator to solve for** $\theta$ To solve for $\theta$, you will need to use the inverse sine function on your calculator. This works backwards by using the ratio of the sides to determine the angle which resulted in that ratio.

Press [shift] [sin] [(] [2] [.] [4] [÷] [3] [)] [=] 53.1301... ≈ 53.13

**Step 3: Write the final answer**

$\theta \approx 53.13°$

NOTE

When you are solving trigonometric equations you might find that you get an error when you try to calculate sin or cos (remember that both the sine and cosine functions have a maximum value of $1$). For these cases there is no solution to the equation.

## WORKED EXAMPLE 10: SOLVING TRIGNOMETRIC EQUATIONS

Question

Solve for $\alpha : 3 \sec \alpha = 1.4$

Solution

**Step 1: Convert sec to cos**

There is no “sec” button on the calculator and so we need to convert sec to cos so we can find $\alpha$.

$$\begin{aligned} 3 \sec \alpha &= 1.4 \\ \frac{3}{\cos \alpha} &= 1.4 \end{aligned}$$

**Step 2: Rearrange the equation**

We need to rearrange the equation so that we have $\cos \alpha$ on one side of the equation.

$$\begin{aligned} \frac{3}{\cos \alpha} &= 1.4 \\ 3 &= 1.4 \cos \alpha \\ \frac{3}{1.4} &= \cos \alpha \end{aligned}$$

**Step 3: Use your calculator to solve for $\alpha$**

To solve for $\alpha$, you will need to use the inverse cosine function on your calculator. This works backwards by using the ratio of the sides to determine the angle which resulted in that ratio.

Press [shift] [cos] [3] [÷] [1] [.] [4] [)] [=] *math error*

In this case we get an error when we try to do the calculation. This is because $\frac{3}{1.4}$ is greater than $1$ and the maximum value of the cosine function is $1$. Therefore there is no solution. It is important in this case to write no solution and not math error.

**Step 4: Write the final answer**

There is no solution

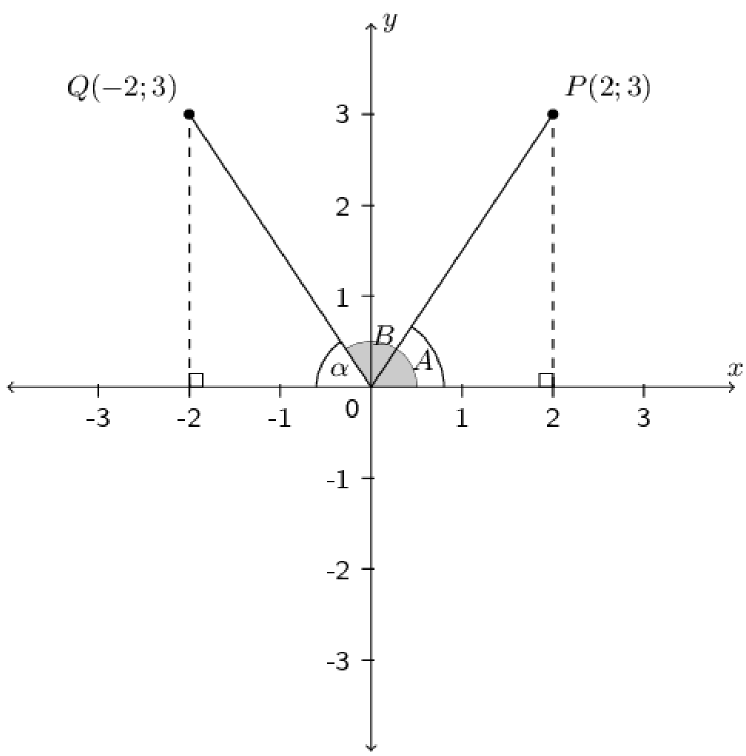
# 9 DEFINING RATIOS IN THE CARTESIAN PLANE

We have defined the trigonometric ratios using right-angled triangles. We can extend these definitions to any angle, noting that the definitions do not rely on the lengths of the sides of the triangle, but on the size of the angle only. So if we plot any point on the Cartesian plane and then draw a line from the origin to that point, we can work out the angle between the $x$-axis and that line. We will first look at this for two specific points and then look at the more general case.

**Finding an angle for specific points**

In the figure below points $P$ and $Q$ have been plotted. A line from the origin ($O$) to each point is drawn. The dotted lines show how we can construct right-angled triangles for each point. The dotted line must always be drawn to the $x$-axis. Now we can find the angles $A$ and $B$:

> NOTE
>
> We can also extend the definitions of the reciprocals in the same way.



From the coordinates of $P(2;3)$, we can see that $x = 2$ and $y = 3$. Therefore, we know the length of the side opposite $\hat{A}$ is $3$ and the length of the adjacent side is $2$. Using:

$$\tan \hat{A} = \frac{opposite}{adjacent} = \frac{3}{2}$$

We calculate that $\hat{A} = 56.3°$

We can also use the theorem of Pythagoras to calculate the hypotenuse of the triangle and then calculate $\hat{A}$ using:

$$\sin \hat{A} = \frac{opposite}{hypotenuse} \text{ or } \cos \hat{A} = \frac{adjacent}{hypotenuse}$$

Consider point $Q(-2;3)$. We define $\hat{B}$ as the angle formed between line $OQ$ and the positive $x$-axis. This is called the standard position of an angle. Angles are always measured from the positive $x$-axis in an anti-clockwise direction. Let $\alpha$ be the angle formed between the line $OQ$ and the negative $x$-axis such that $\hat{B} + \alpha = 180°$.

From the coordinates of $Q(-2;3)$, we know the length of the side opposite $\alpha$ is $3$ and the length of the adjacent side is $2$. Using:

$$\tan \alpha = \frac{opposite}{adjacent} = \frac{3}{2}$$

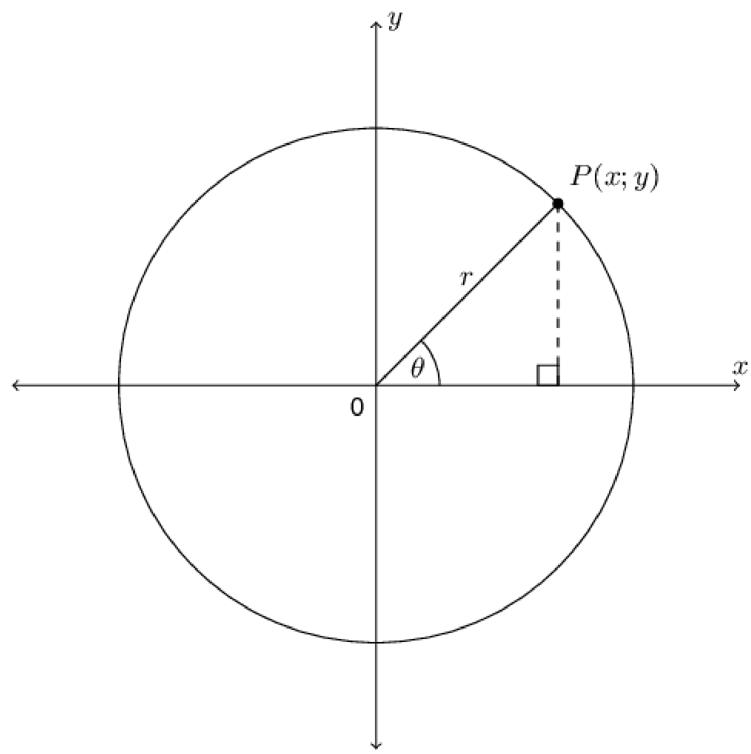
we calculate that $\alpha = 56.3°$. Therefore $\hat{B} = 180° - \alpha = 123.7°$.

Similarly, an alternative method is to calculate the hypotenuse using the theorem of Pythagoras and calculate $\alpha$ using:

$$\sin \alpha = \frac{opposite}{hypotenuse} \text{ or } \cos \alpha = \frac{adjacent}{hypotenuse}$$

**Finding any angle**

If we were to draw a circle centred on the origin ($O$) and passing through the point $P(x; y)$, then the length from the origin to point $P$ is the radius of the circle, which we denote $r$. We denote the angle formed between the line $OP$ and the $x$-axis as $\theta$.

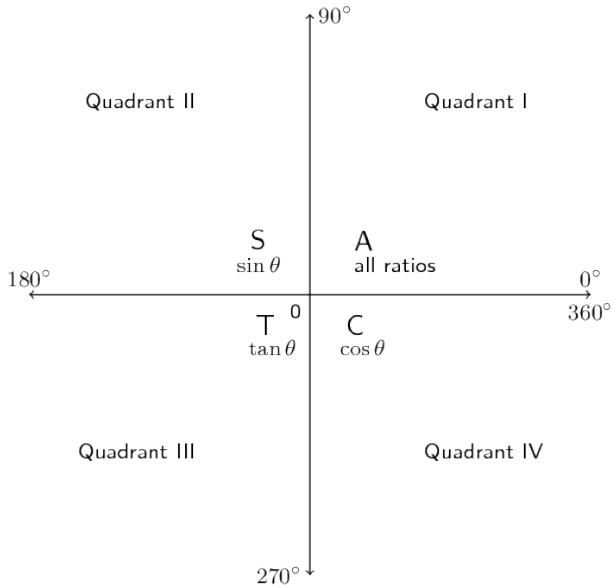



We can rewrite all the trigonometric ratios in terms of $x, y$ and $r$. The general definitions for the trigonometric ratios are:

$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

**The CAST diagram**

The Cartesian plane is divided into 4 quadrants in an anti-clockwise direction as shown in the diagram below. Notice that $r$ is always positive but the values of $x$ and $y$ change depending on the position of the point in the Cartesian plane. As a result, the trigonometric ratios can be positive or negative. The letters $C, A, S$ and $T$ indicate which of the ratios are positive in each quadrant:




This diagram is known as the CAST diagram.

We note the following using the general definitions of the trigonometric ratios:

- Quadrant I

  Both the $x$ and $y$ values are positive so all ratios are positive in this quadrant.

- Quadrant II

  The $y$ values are positive therefore sin and csc are positive in this quadrant (recall that sin and csc are defined in terms of $y$ and $r$).

- Quadrant III

  Both the $x$ and the $y$ values are negative therefore tan and cot are positive in this quadrant (recall that tan and cot are defined in terms of $x$ and $y$).

- Quadrant IV

  The $x$ values are positive therefore cos and sec are positive in this quadrant (recall that cos and sec are defined in terms of $x$ and $r$).

> **IMPORTANT**
>
> The hypotenuse, $r$, is a length, and is therefore always positive.

**Special angles in the Cartesian plane**

When working in the Cartesian plane we include two other special angles in right-angled triangles: $0°$ and $90°$.

Notice that when $\theta = 0°$ the length of the opposite side is equal to $0$ and the length of the adjacent side is equal to the length of the hypotenuse. Therefore:

$$\begin{aligned} \sin 0° &= \frac{opposite}{hypotenuse} = \frac{0}{hypotenuse} = 0 \\ \cos 0° &= \frac{adjacent}{hypotenuse} = \frac{hypotenuse}{hypotenuse} = 1 \\ \tan 0° &= \frac{opposite}{adjacent} = \frac{0}{adjacent} = 0 \end{aligned}$$

When $\theta = 90°$ the length of the adjacent side is equal to 0 and the length of the opposite side is equal to the length of the hypotenuse. Therefore:

$$\begin{aligned} \sin 90° &= \frac{opposite}{hypotenuse} = \frac{hypotenuse}{hypotenuse} = 1 \\ \cos 0° &= \frac{adjacent}{hypotenuse} = \frac{0}{hypotenuse} = 0 \\ \tan 0° &= \frac{opposite}{adjacent} = \frac{opposite}{0} = undefined \end{aligned}$$

Now we can extend our knowledge of special angles.

| $\theta$ | $30°$ | $30°$ | $45°$ | $60°$ | $90°$ |
|---|---|---|---|---|---|
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | undefined |

### WORKED EXAMPLE 11: RATIOS IN THE CARTESIAN PLANE

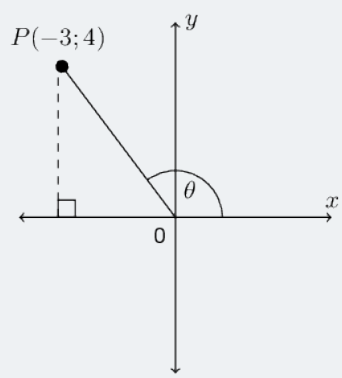
Question

$P(-3;4)$ is a point on the Cartesian plane with origin $O$.   is the angle between $OP$ and the positive $x$-axis. Without using a calculator, determine the value of:

1. $\cos\theta$
2. $3\tan\theta$
3. $\frac{1}{2}\csc\theta$

Solution

**Sketch point $P$ in the Cartesian plane and label the angle $\theta$**




**Use the theorem of Pythagoras to calculate $r$**

$$
\begin{aligned}
r^2 &= x^2 + y^2 \\
&= (-3)^2 + (4)^2 \\
&= 25 \\
\therefore \quad r &= 5
\end{aligned}
$$

Note $r$ is positive as it is the radius of the circle.

WORKED EXAMPLE 11: RATIOS IN THE CARTESIAN PLANE (CONTINUED)

**Step 3: Substitute values for $x, y$ and $r$ into the required ratios**

We note that $x = -3$, $y = 4$ and $r = 5$.

1. $\cos\theta = \frac{x}{r} = \frac{-3}{5}$
2. $3\tan\theta = 3(\frac{y}{x}) = 3(\frac{4}{-3}) = -4$
3. $\frac{1}{2}\csc\theta = \frac{1}{2}(\frac{r}{y}) = \frac{1}{2}(\frac{5}{4}) = \frac{5}{8}$

## WORKED EXAMPLE 12: RATIOS IN THE CARTESIAN PLANE

Question

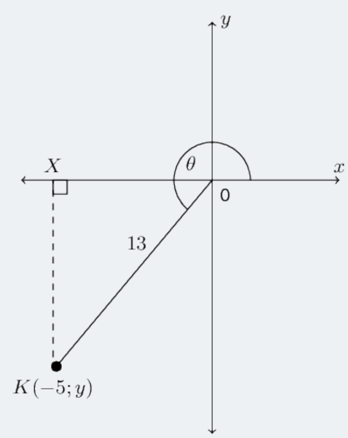
$X\hat{O}K = \theta$ is an angle in the third quadrant where $X$ is a point on the positive $x$-axis and $K$ is the point $(-5; y)$. $OK$ is $13$ units.

1. Determine, without using a calculator, the value of $y$.

2. Prove that $\tan^2\theta + 1 = \sec^2\theta$ without using a calculator.

Solution

**Step 1: Sketch point $K$ in the Cartesian plane and label the angle $\theta$**




**Step 2: Use the theorem of Pythagoras to calculate $y$**

$$\begin{aligned} r^2 &= x^2 + y^2 \\ y^2 &= r^2 - x^2 \\ &= (13)^2 - (-5)^2 \\ &= 169 - 25 \\ &= 144 \\ y &= \pm 12 \end{aligned}$$

Given that $\theta$ lies in the third quadrant, $y$ must be negative.

$$\therefore \quad y = -12$$

WORKED EXAMPLE 12: RATIOS IN THE CARTESIAN PLANE (CONTINUED)

**Step 3: Substitute values for $x, y$ and $r$ and simplify**

$x = -5, y = -12$ and $r = 13$

LHS

$$\begin{aligned} \tan^2\theta + 1 &= (\frac{y}{x})^2 + 1 \\ &= (\frac{-12}{-5})^2 + 1 \\ &= (\frac{144}{25})^2 + 1 \\ &= \frac{144 + 25}{25} \\ &= \frac{169}{25} \end{aligned}$$

RHS

$$\begin{aligned} \sec^2\theta &= (\frac{r}{x})^2 \\ &= (\frac{13}{-5})^2 \\ &= \frac{169}{25} \end{aligned}$$

Therefore the LHS = RHS and we have proved that $\tan^2\theta + 1 = \sec^2\theta$.

NOTE

Whenever you have to solve trigonometric problems without a calculator, it can be very helpful to make a sketch.

# 10 CHAPTER SUMMARY

- We can define three trigonometric ratios for right-angled triangles: sine (sin), cosine (cos) and tangent (tan)

  $\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r}$

  $\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r}$

  $\tan\theta = \frac{opposite}{adjacent} = \frac{y}{x}$

- Each of these ratios have a reciprocal: cscant (csc ), secant (sec) and cotangent (cot)

  csc $\theta$ = $\frac{hypotenuse}{opposite} = \frac{r}{y}$

  sec $\theta$ = $\frac{hypotenuse}{adjacent} = \frac{r}{x}$

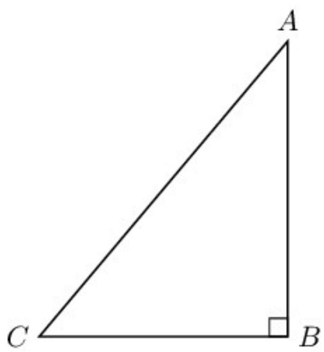
  cot $\theta$ = $\frac{adjacent}{opposite} = \frac{x}{y}$

- We can use the principles of solving equations and the trigonometric ratios to help us solve simple trigonometric equations.
- For some special angles ($0°$, $30°$, $45°$, $60°$ and $90°$), we can easily find the values of sin, cos and tan without using a calculator.
- We can extend the definitions of the trigonometric ratios to any angle.

# 11 EXERCISES

## 11.1 Exercise 1

1. Complete each of the following:




1.1 $\sin \hat{A}$

1.2 $\cos \hat{A}$

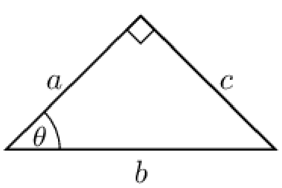
1.3 $\tan \hat{A}$

1.4 $\sin \hat{C}$

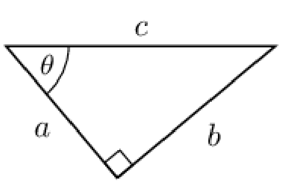
1.5 $\cos \hat{C}$

1.6 $\tan \hat{C}$

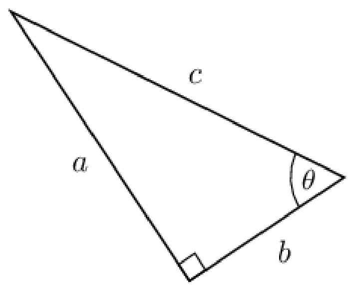
2. In each of the following triangles, state whether $a$, $b$ and $c$ are the hypotenuse, opposite or adjacent sides of the triangle with respect to $\theta$.

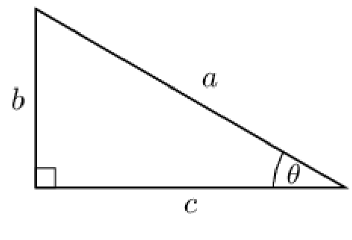


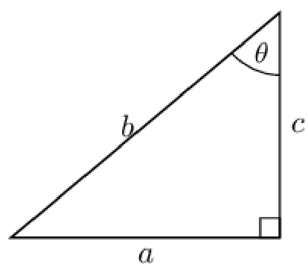

2.1

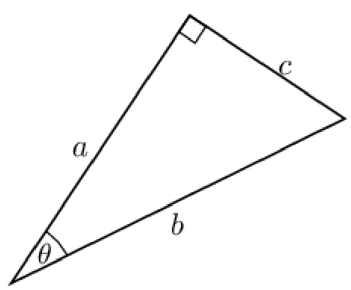



2.2

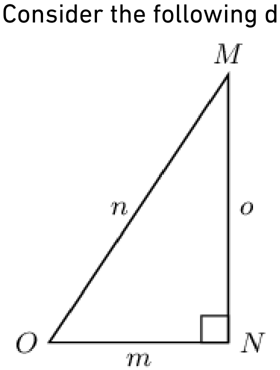
2.3 


2.4 


2.5 


2.6 


3. Consider the following diagram:



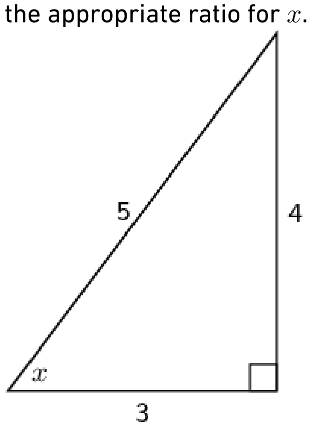

3.1 Write down $\cos \hat{O}$ in terms of $m$, $n$ and $o$.

3.2 Write down tan $\hat{M}$ in terms of $m$, $n$ and $o$.

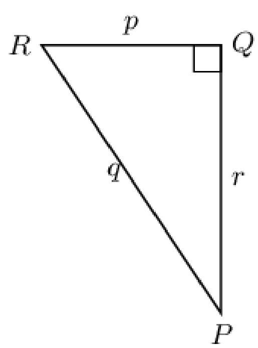
3.3 Write down sin $\hat{O}$ in terms of $m$, $n$ and $o$.

3.4 Write down cos $\hat{M}$ in terms of $m$, $n$ and $o$.

4. Find $x$ in the diagram in three different ways. You do not need to calculate the value of $x$, just write down the appropriate ratio for $x$.



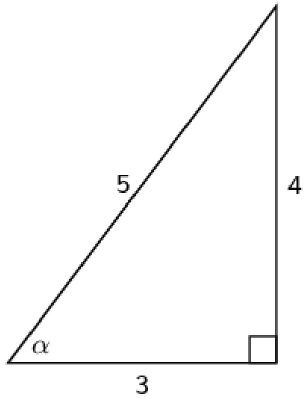

5. Which of these statements is true about $\Delta PQR$?




- sin $\hat{R} = \frac{p}{q}$
- tan $\hat{Q} = \frac{r}{p}$
- cos $\hat{P} = \frac{r}{q}$
- sin $\hat{P} = \frac{p}{r}$

6. Sarah wants to find the value of $\alpha$ in the triangle below.

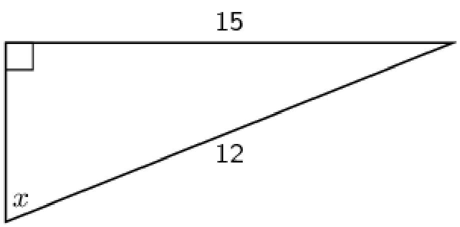
   Which statement is a correct line of working?



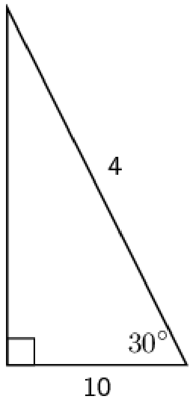

- $\sin\alpha = \frac{4}{5}$
- $\cos\left(\frac{3}{5}\right) = \alpha$
- $\tan\alpha = \frac{5}{4}$

7. Explain what is wrong with each of the following diagrams.

7.1




7.2



## 11.2 Exercise 2

1. Use your calculator to determine the value of the following (correct to 2 decimal places):

    1.1 $\tan 65°$

    1.2 $\sec 10°$

    1.3 $\sec 48°$

    1.4 $\cot 32°$

    1.5 $\csc 140°$

    1.6 $\csc 237°$

    1.7 $\sec 231°$

    1.8 $\csc 226°$

    1.9 $\frac{1}{4}\cos 20°$

    1.10 $3\tan 40°$

    1.11 $\frac{2}{3}\sin 90°$

    1.12 $\sin 38°$

    1.13 $\frac{5}{\cos 4{,}3°}$

    1.14 $\sqrt{\sin 55°}$

    1.15 $\frac{\sin 90°}{\cos 90°}$

    1.16 $\tan 35° + \cot 35°$

    1.17 $\frac{2+\cos 310°}{2+\sin 87°}$

    1.18 $\sqrt{4\sec 99°}$

    1.19 $\sqrt{\frac{\cot 103°+\sin 1090°}{\sec 10°+5}}$

    1.20 $\cos 74°$

    1.21 $\sin 12°$

    1.22 $\cos 26°$

    1.23 $\tan 49°$

    1.24 $\sin 305°$

    1.25 $\tan 124°$

    1.26 $\sec 65°$

2. If $x = 39°$ and $y = 21°$, use a calculator to determine whether the following statements are true or false:

    2.1 $\cos x + 2\cos x = 3\cos x$

2.2 $\cos 2y = \cos y + \cos y$

2.3 $\tan x = \frac{\sin x}{\cos x}$

2.4 $\cos(x + y) = \cos x + \cos y$

3. Solve for $x$ in $5^{\tan x} = 125$

## 11.3 Exercise 3

1. Select the answer for each expression:

1.1 $\cos 45°$

1.2 $\sin 45°$

1.3 $\tan 30°$

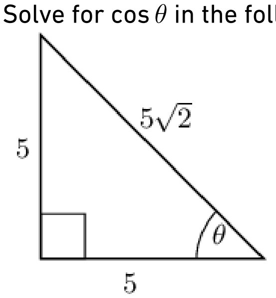
1.4 $\tan 60°$

1.5 $\cos 45°$

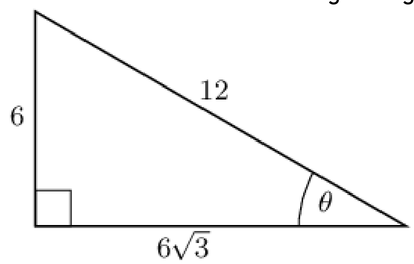
1.6 $\tan 49°$

1.7 $\cos 60°$

2. Solve for $\cos \theta$ in the following triangle, in surd form:



3. Solve for $\tan\theta$ in the following triangle, in surd form:




4. Calculate the value of the following without using a calculator:

   4.1 $\sin 45° \times \cos 45°$

   4.2 $\cos 60° + \tan 45°$

   4.3 $\sin 60° - \cos 60°$

5. Evaluate the following without using a calculator.

   5.1 $\tan 45° \div \sin 60°$

   5.2 $\tan 30° - \sin 60°$

   5.3 $\sin 30° - \tan 45° - \sin 30°$

   5.4 $\tan 30° \div \tan 30° \div \sin 45°$

   5.5 $\sin 45° \div \sin 30° \div \cos 45°$

   5.6 $\tan 60° - \tan 60° - \sin 60°$

   5.7 $\cos 45° - \sin 60° - \sin 45°$

6. Use special angles to show that:

   6.1 $\frac{\sin 60°}{\cos 60°} = \tan 60°$

   6.2 $\sin^2 45° + \cos^2 45° = 1$

   6.3 $\cos 30° = \sqrt{1 - \sin^2 30°}$

7. Use the definitions of the trigonometric ratios to answer the following questions:

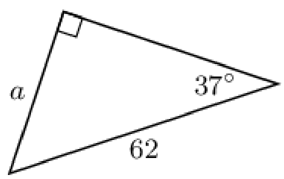
   7.1 In the following equation, what is the value of $x$?:
   sin $\alpha \leq x$ for all values of $\alpha$.

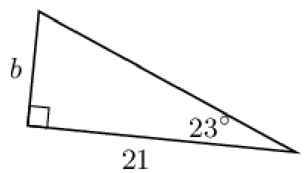
   7.2 Explain why cos $\alpha$ has a maximum value of $1$.

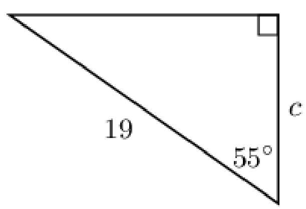
   7.3 Is there a maximum value for tan $\alpha$?

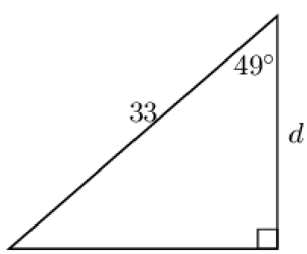
## 11.4 Exercise 4

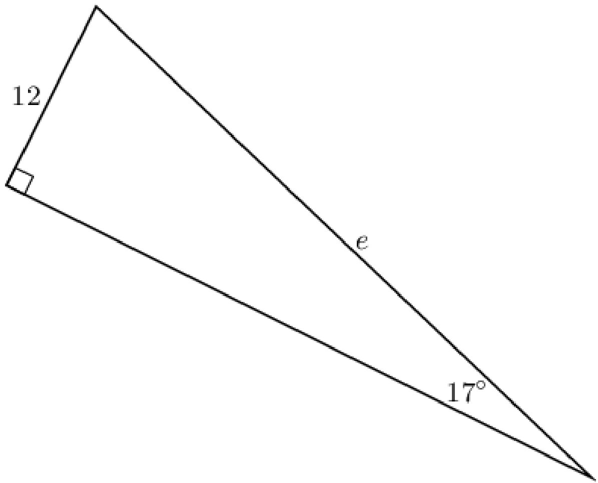
1. In each triangle find the length of the side marked with a letter. Give your answers correct to $2$ decimal places.

1.1


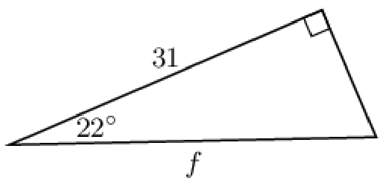

1.2



1.3


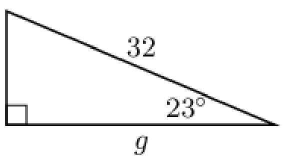

1.4




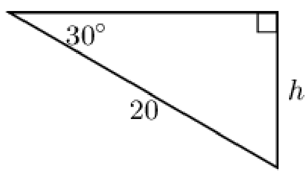

1.5



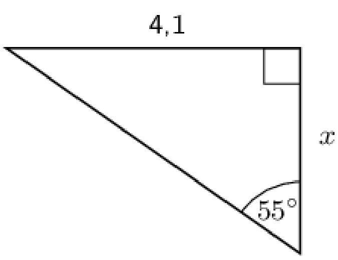

1.6



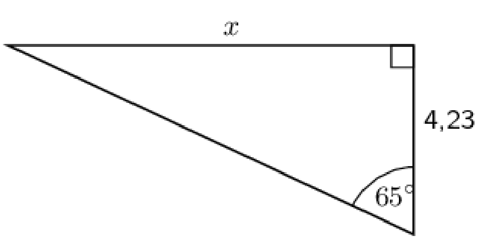

1.7




1.8

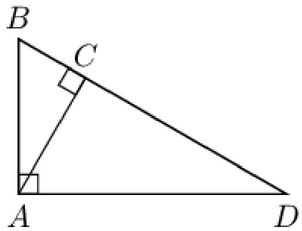



1.9




1.10

2. Write down two ratios for each of the following in terms of the sides: $AB$; $BC$; $BD$; $AD$; $DC$ and $AC$



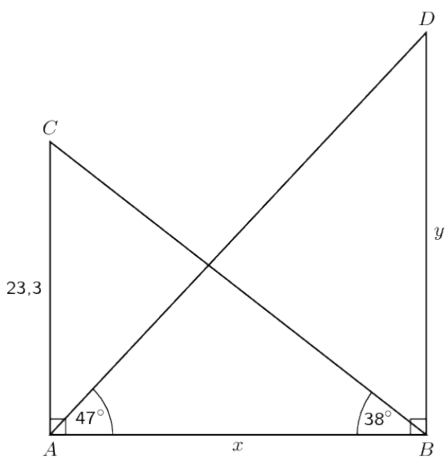

2.1 sin $\hat{B}$

2.2 cos $\hat{D}$

2.3 tan $\hat{B}$

3. In $\Delta MNP$, $\hat{N} = 90°$, $MP = 20$ and $\hat{P} = 40°$. Calculate $NP$ and $MN$ (correct to 2 decimal places).

4. Calculate $x$ and $y$ in the following diagram:




## 11.5 Exercise 5

1. Determine the angle (correct to $1$ decimal place):

1.1 tan $\alpha = 1,7$

1.2 sin $\beta + 2 = 2,65$

1.3 $2$ sin $\theta + 5 = 0.8$

1.4 $3$ tan $\beta = 1$

1.5 sin $3\alpha = 1,2$

1.6 tan $\frac{\theta}{3} =$ sin $48°$

1.7 $\frac{1}{2}$ cos $2\beta = 0,3$

1.8 $2$ sin $3\theta + 1 = 2,6$

1.9 $\sin\theta = 0,8$

1.10 $\cos\alpha = 0,32$

1.11 $\tan\beta = 4,2$

1.12 $\tan\theta = 5\frac{3}{4}$

1.13 $\sin\theta = \frac{2}{3}$

1.14 $\cos\beta = 1,2$

1.15 $4\cos\theta = 3$

1.16 $\cos 4\theta = 0,3$

2. If $x = 16°$ and $y = 36°$, use your calculator to evaluate each of the following, correct to 3 decimal places.

2.1 $\sin(x - y)$

2.2 $3\sin x$

2.3 $\tan x - \tan y$

2.4 $\cos x + \cos y$

2.5 $\frac{1}{3}\tan y$

2.6 $\csc(x - y)$

2.7 $2\cos x + \cos 3y$

2.8 $\tan(2x - 5y)$

3. In each of the following find the value of $x$ correct to two decimal places.

3.1 $\sin x = 0,814$

3.2 $\sin x = \tan 45°$

3.3 $\tan 2x = 3,123$

3.4 $\tan x = 3\sin 41°$

3.5 $\sin(2x + 45°) = 0,123$

3.6 $\sin(x - 10°) = \cos 57°$

# 12 ANSWERS FOR EXERCISES

## 12.1 Exercise 1

1. 1.1 $\frac{CB}{AC}$
   1.2 $\frac{AB}{AC}$
   1.3 $\frac{CB}{AB}$
   1.4 $\frac{AB}{AC}$
   1.5 $\frac{CB}{AC}$
   1.6 $\frac{AB}{CB}$
2. 2.1 a=adjacent, b=hypotenuse, c=opposite
   2.2 a=adjacent, b=opposite, c=hypotenuse
   2.3 a=opposite, b=adjacent, c=hypotenuse
   2.4 a=hypotenuse, b=opposite, c=adjacent
   2.5 a=opposite, b=hypotenuse, c=adjacent
   2.6 a=adjacent, b=hypotenuse, c=opposite
3. 3.1 $\frac{m}{n}$
   3.2 $\frac{m}{o}$
   3.3 $\frac{o}{n}$
   3.4 $\frac{m}{o}$
4. $\sin x = \frac{4}{5}$
   $\cos x = \frac{3}{5}$
   $\tan x = \frac{4}{3}$
5. $\cos \hat{P} = \frac{r}{q}$
6. $\sin \alpha = \frac{4}{5}$
7. 7.1 The Hypotenuse of a right angled triangle cannot smaller in length than any of the sides
   7.2 The Hypotenuse of a right angled triangle cannot smaller in length than any of the sides

## 12.2 Exercise 2

1. 1.1 $2{,}14$
   1.2 $1{,}02$
   1.3 $1{,}49$
   1.4 $1{,}60$
   1.5 $1{,}56$
   1.6 $-1{,}19$
   1.7 $-1{,}59$
   1.8 $-1{,}39$
   1.9 $0{,}23$
   1.10 $2{,}52$
   1.11 $0{,}67$
   1.12 $0{,}62$
   1.13 $5{,}01$
   1.14 $0{,}91$
   1.15 undefined
   1.16 $2{,}13$
   1.17 $0{,}88$
   1.18 Non-real
   1.19 $0{,}21$
   1.20 $0{,}28$
   1.21 $0{,}21$
   1.22 $0{,}90$
   1.23 $1{,}15$
   1.24 $-0{,}82$
   1.25 $-1{,}48$
   1.26 $2{,}37$
2. 2.1 True
   2.2 False
   2.3 True
   2.4 False
3. $x = 71{,}57$

## 12.3 Exercise 3

1. 1.1 $\frac{1}{\sqrt{2}}$

   1.2 $\frac{1}{\sqrt{2}}$

   1.3 $\frac{1}{\sqrt{3}}$

   1.4 $\frac{\sqrt{3}}{1}$

   1.5 $\frac{1}{\sqrt{2}}$

   1.6 $1{,}15$

   1.7 $\frac{1}{2}$

2. $\frac{1}{\sqrt{2}}$

3. $\frac{1}{\sqrt{3}}$

4. 4.1 $\frac{1}{2}$

   4.2 $\frac{3}{2}$

   4.3 $\frac{\sqrt{3}-1}{2}$

5. 5.1 $\frac{2}{\sqrt{3}}$

   5.2 $\frac{-1}{2\sqrt{3}}$

   5.3 $-1$

   5.4 $\frac{\sqrt{2}}{1}$

   5.5 $2$

   5.6 $\frac{-\sqrt{3}}{2}$

   5.7 $\frac{-\sqrt{3}}{2}$

6. 6.1 $LHS = RHS = \sqrt{3}$

   6.2 $LHS = RHS = 1$

   6.3 $LHS = RHS = \frac{\sqrt{3}}{2}$

7. 7.1 $x = \frac{\text{opposite}}{\text{hypotenuse}} \leq 1$

   7.2 $\frac{\text{hypotenuse}}{\text{hypotenuse}} = 1$

   7.3 Ratio = $\frac{\text{opposite}}{\text{adjacent}}$ so there is no maximum value for tan $\alpha$

## 12.4 Exercise 4

1. 1.1 $37,31$
   1.2 $8,91$
   1.3 $10,90$
   1.4 $21,65$
   1.5 $41,04$
   1.6 $33,43$
   1.7 $29,46$
   1.8 $10,00$
   1.9 $2,87$
   1.10 $9,07$
2. 2.1 $\frac{AC}{AB} = \frac{AD}{BD}$
   2.2 $\frac{AD}{BD} = \frac{CD}{AD}$
   2.3 $\frac{AC}{BC} = \frac{AD}{AB}$
3. $MN = 12{,}86$ and $NP = 15{,}32$
4. $x = 29{,}82$ and $y = 31{,}98$

## 12.5 Exercise 5

1. 1.1 $59{,}5°$
   1.2 $40{,}5°$
   1.3 no solution
   1.4 $18{,}4°$
   1.5 no solution
   1.6 $109{,}9°$
   1.7 $26{,}6°$
   1.8 $17{,}7°$
   1.9 $53{,}1°$
   1.10 $71{,}3°$
   1.11 $76{,}6°$
   1.12 $80{,}1°$

1.13 $41,8°$

1.14 No solution

1.15 $41,4°$

1.16 $18,1°$

2. 2.1 $-0,342$

   2.2 $0,827$

   2.3 $-0,440$

   2.4 $1,770$

   2.5 $0,242$

   2.6 $-2,924$

   2.7 $1,614$

   2.8 $0,625$

3. 3.1 $54,49°$

   3.2 $90°$

   3.3 $36,12°$

   3.4 $63,07°$

   3.5 $-18,97°$

   3.6 $43°$