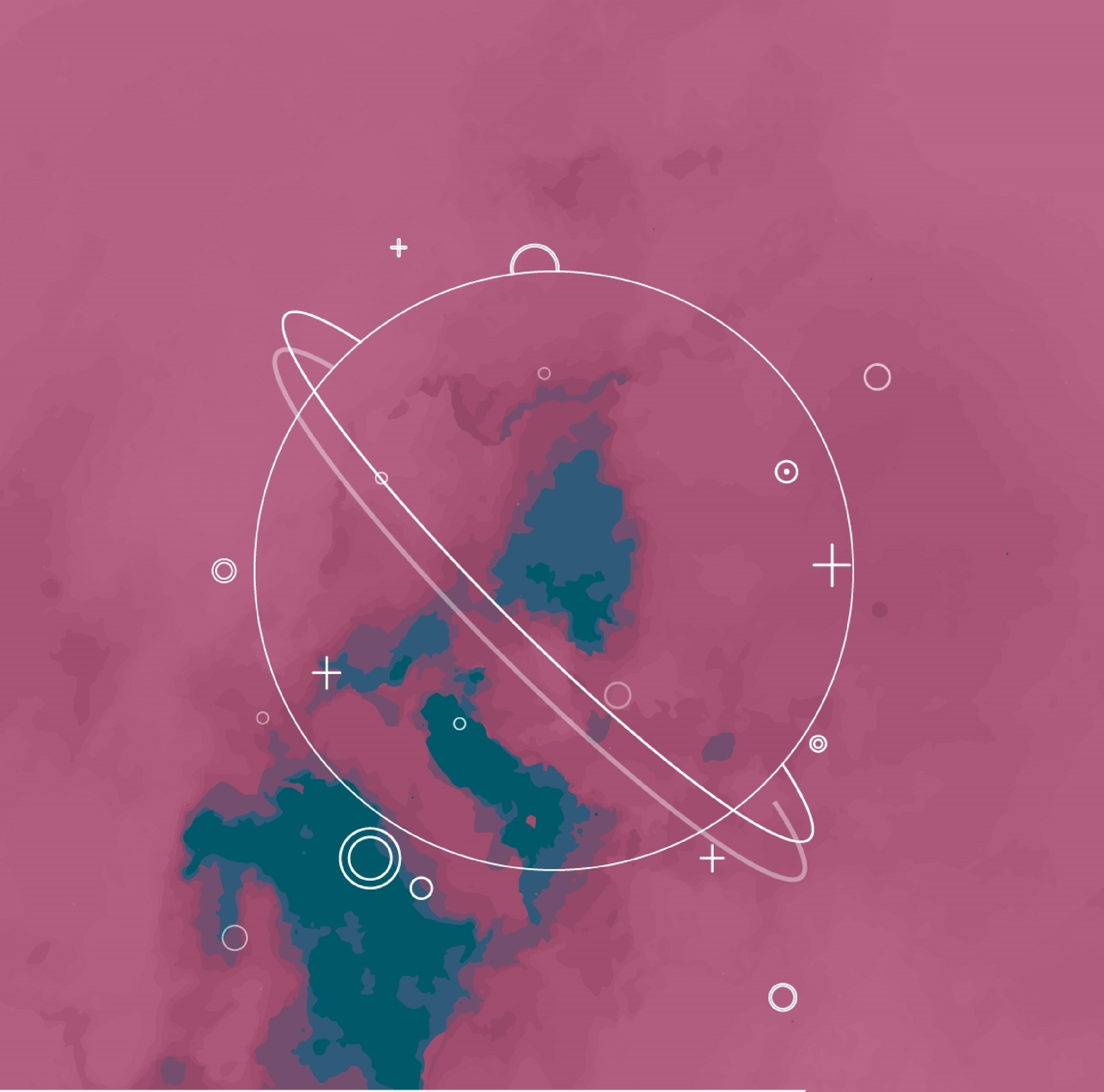

CHAPTER 7

Euclidean Geometry

CONTENTS

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MathU Teaching® Emporium

1 INTRODUCTION: EUCLIDEAN GEOMETRY

Geometry (from the Greek "geo" = earth and "metria" = measure) arose as the field of knowledge dealing with spatial relationships. Analytical geometry deals with space and shape using algebra and a coordinate system. Euclidean geometry deals with space and shape using a system of logical deductions.

Euclidean geometry was first used in surveying and is still used extensively for surveying today. Euclidean geometry is also used in architecture to design new buildings. Other uses of Euclidean geometry are in art and to determine the best packing arrangement for various types of objects.

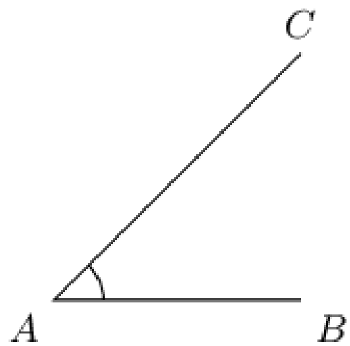
Figure 1: A small piece of the original version of Euclid's elements. Euclid is considered to be the father of modern geometry. Euclid's elements was used for many years as the standard text for geometry.

DID YOU KNOW?

In Euclidean geometry we use two fundamental types of measurement: angles and distances.

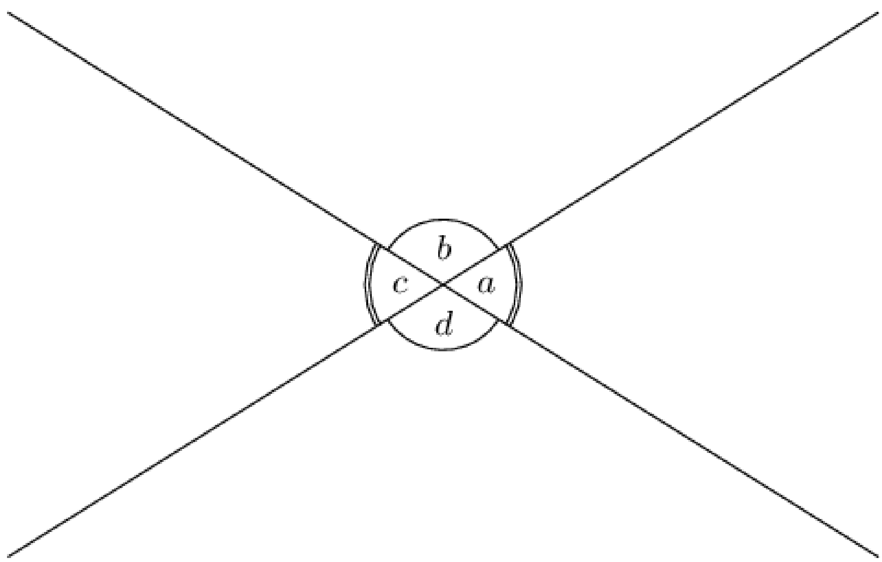
2 ANGLES

An angle is formed when two straight lines meet at a point, also known as a vertex. Angles are labelled with a caret on a letter, for example, $\hat{B}$. Angles can also be labelled according to the line segments that make up the angle, for example $C\hat{B}A$ or $A\hat{B}C$. The $\angle$ symbol is a short method of writing angle in geometry and is often used in phrases such as "sum of $\angle$s in $\triangle$". Angles are measured in degrees which is denoted by $^\circ$, a small circle raised above the text, similar to an exponent.



3 PROPERTIES AND NOTATION

In the diagram below two straight lines intersect at a point, forming the four angles $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$.



The following table summarises the different types of angles, with examples from the figure above.

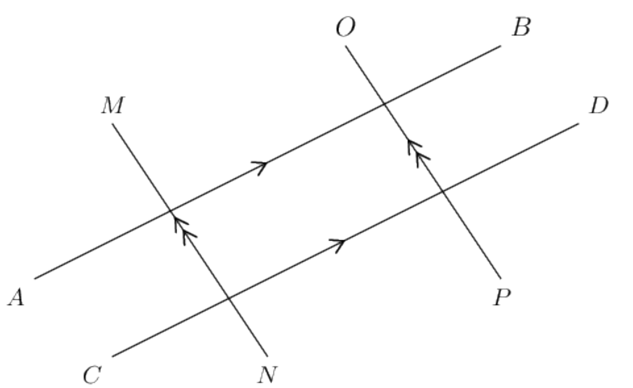
Term	Property	Examples
Acute angle	$0^\circ <$ angle $< 90^\circ$	$\hat{a}; \hat{c}$
Right angle	Angle $= 90^\circ$	
Obtuse angle	$90^\circ <$ angle $< 180^\circ$	$\hat{b}; \hat{d}$
Straight angle	Angle $= 180^\circ$	$\hat{a} + \hat{b}; \bar{b} + \hat{c}$
Reflex angle	$180^\circ <$ angle $< 360^\circ$	$\hat{a} + \hat{b} + \hat{c}$
Adjacent angles	Angles that share a vertex and a common side.	$\hat{a}$ and $\hat{d}$; $\hat{c}$ and $\hat{d}$
Vertically opposite angles	Angles opposite each other when two lines intersect. They share a vertex and are equal.	$\hat{a} = \hat{c}; \hat{b} = \hat{d}$
Supplementary angles	Two angles that add up to 180°	$\hat{a} + \hat{b} = 180^\circ$ $\hat{b} + \hat{c} = 180^\circ$
Complementary angles	Two angles that add up to 90°	
Revolution	The sum of all angles around a point	$\hat{a} + \hat{b} + \hat{c} + \hat{d} = 360^\circ$

Note that adjacent angles on a straight line are supplementary.

4 PARALLEL LINES AND TRANSVERSAL LINES

Two lines intersect if they cross each other at a point. For example, at a traffic intersection two or more streets intersect; the middle of the intersection is the common point between the streets.

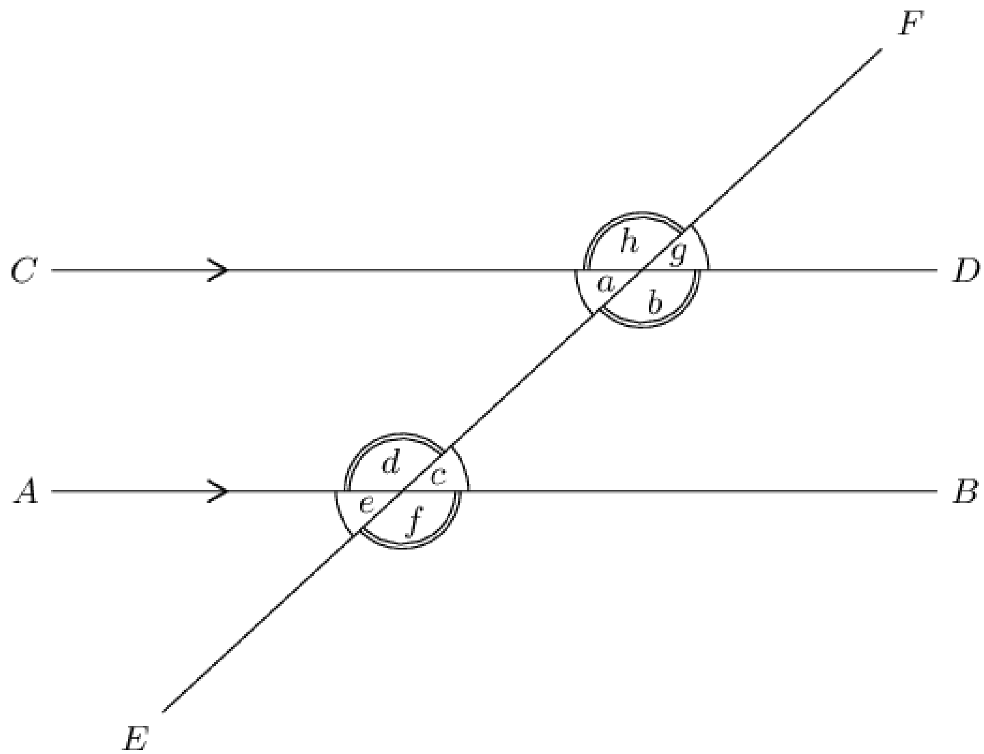
Parallel lines are always the same distance apart and they are denoted by arrow symbols as shown below.



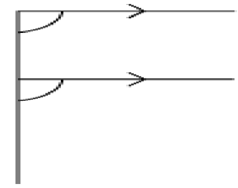
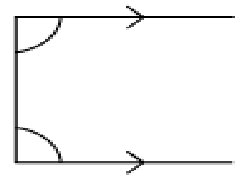
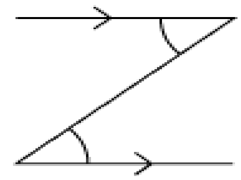
In writing we use two vertical lines to indicate that two lines are parallel:

$$AB \parallel CD \text{ and } MN \parallel OP$$

A transversal line intersects two or more parallel lines. In the diagram below, $AB \parallel CD$ and EF is a transversal line,


F
h
g
C
D
a
b
d
c
A
B
e
f
E

The properties of the angles formed by these intersecting lines are summarised in the following table:

Name of angle	**Definition**	**Examples**	**Notes**
Interior Angles	Angles that lie in between the parallel lines.	$\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are interior angles	Interior means inside
Exterior angles	Angles that lie outside the parallel lines	$\hat{e}, f, \hat{g},$ and $\hat{h}$ are exterior angles	Exterior means outside
Corresponding angles	Angles on the same side of the lines and the same side of the transversal. If the lines are parallel, the corresponding angles will be equal.	$\hat{a}$ and $\hat{e}, \hat{b}$ and $f, \hat{c}$ and $\hat{g}, \hat{d}$ and $\hat{h}$ are pairs of corresponding angles. $\hat{a} = \hat{e}, \hat{b} = f, \hat{c} = \hat{g}$ and $\hat{d} = \hat{h}$	 F Shape
Co-interior angles	Angles that lie in between the lines and on the same side of the transversal. If the lines are parallel, the angles are supplementary.	$\hat{a}$ and $\hat{d}, \hat{b}$ and $\hat{c}$ are pairs of co-interior angles. $\hat{a} + \hat{d} = 180°$ $\hat{b} + \hat{c} = 180°$	 C Shape
Alternate angles	Equal interior angles that lie inside the lines and on opposite sides of the transversal. If the lines are parallel, the interior angles will be equal.	$\hat{a}$ and $\hat{c}, \hat{b}$ and d are pairs of alternate interior angles. $\hat{a} = \hat{c}, \hat{b} = \hat{d}$	 Z Shape

If two lines are intersected by a transversal such that:

- corresponding angles are equal; or
- alternate interior angles are equal; or
- co-interior angles are supplementary

then the two lines are parallel.

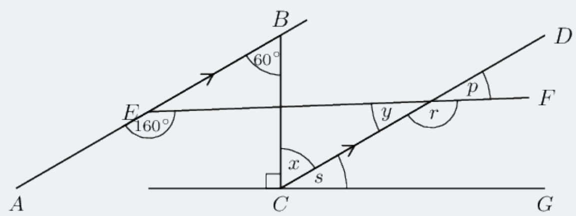
NOTE

When we refer to lines we can either write EF to mean the line through points E and F or $\overline{EF}$ to mean the line segment from point E to point F.

WORKED EXAMPLE 1: FINDING ANGLES

Question

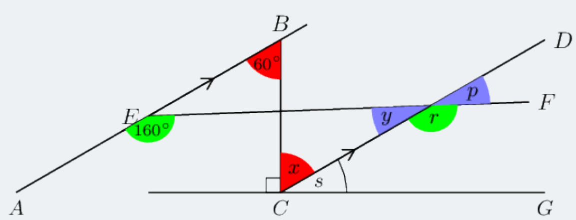
Find all the unknown angles. Is $EF \parallel CG$? Explain your answer.



Solution

Step 1: Use the properties of parallel lines to find all equal angles on the diagram

Redraw the diagram and mark all the equal angles.



Step 2: Determine the unknown angles

$$AB \parallel CD \quad (given)$$

$$\therefore \hat{x} = 60^\circ \quad (alt; AB \parallel CD)$$

$$\hat{y} + 160^\circ = 180^\circ \quad (co-int; AB \parallel CD)$$

$$\therefore \hat{y} = 20^\circ$$

$$\hat{p} = \hat{y} \quad (vertopp =)$$

$$\therefore \quad \hat{p} = 60^\circ$$

$$\hat{r} = 160^\circ \quad (corresp; AB \parallel CD)$$

WORKED EXAMPLE 1: FINDING ANGLES (CONTINUED)

$$\hat{s} + \hat{x} + 90° = 180° \text{ (on a str line)}$$
$$\hat{s} + 60° = 90°$$
$$\therefore \quad \hat{s} = 30°$$

Step 3: Determine whether $EF \equiv CG$

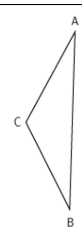
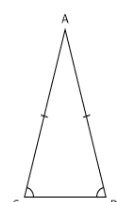
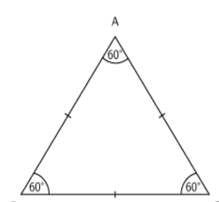
if $EF \equiv CG$ then $\hat{p}$ will be equal to corresponding angle $\hat{s}$, but $\hat{p}$ = $20°$ and $\hat{s}$ = $30°$.

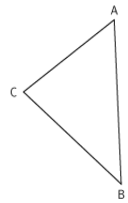
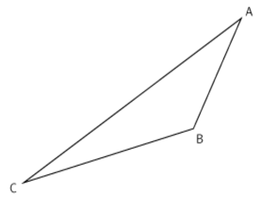
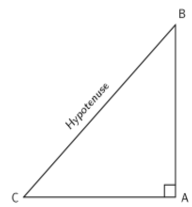
$\therefore$ EF is not parallel to CG.

5 TRIANGLES

Classification of triangles

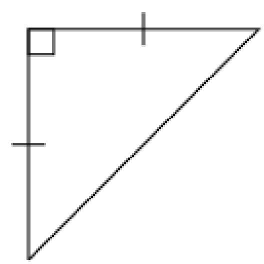
A triangle is a three-sided polygon. Triangles can be classified according to sides: equilateral, isosceles and scalene. Triangles can also be classified according to angles: acute-angled, obtuse-angled and right-angled. We use the notation $\triangle ABC$ to refer to a triangle with vertices labelled A, B and C.

Name	Diagram	Properties
Scalene		All sides and angles are different
Isoceles		Two sides are equal in length. The angles opposite the equal sides are also equal
Equilateral		All three sides are equal in length and all three angles are equal

Acute		Each of the three interior angles is less than $90°$
Obtuse		One interior angles is greater than $90°$
Right-angled		One interior angle is $90°$

Different combinations of these properties are also possible. For example, an obtuse isosceles triangle and a right-angled isosceles triangle are shown below:

obtuse isosceles

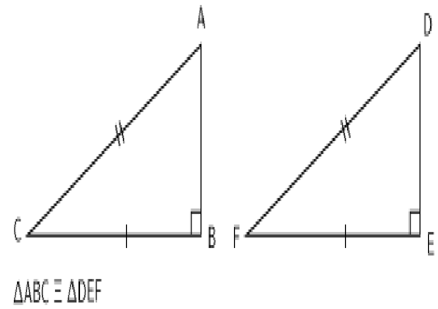
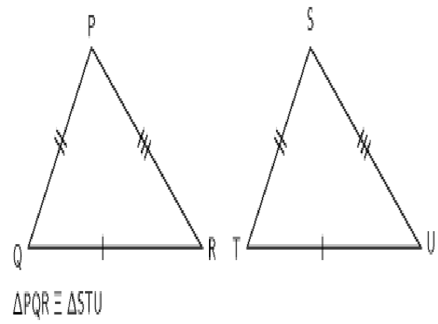
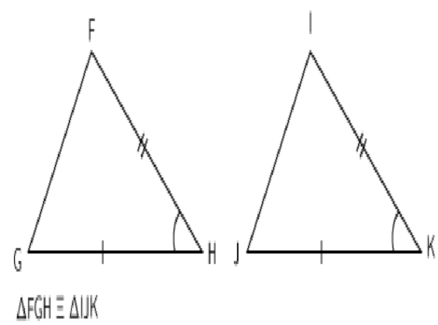
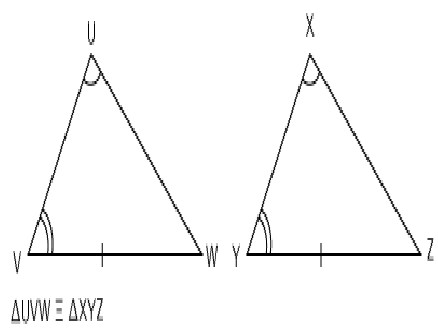


right-angled isosceles

Congruency

Two triangles are congruent if one fits exactly over the other. This means that the triangles have equal corresponding angles and sides. To determine whether two triangles are congruent, it is not necessary to check every side and every angle. We indicate congruency using $\equiv$.

The following table describes the requirements for congruency:

Rule	Description	Diagram
RHS or $90°$HS $(90°,$ hypotenuse, side)	If the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the corresponding side of another right-angled triangle, then the two triangles are congruent.	
SSS (side, side, side)	If three sides of a triangle are equal in length to the corresponding sides of another triangle, then the two triangles are congruent.	
SAS or $S\angle S$ (side, angle, side)	If two sides and the **included** angle of a triangle are equal to the corresponding two sides and **included** angle of another triangle, then the two triangles are congruent.	
RHS or AAS or $\angle\angle S$ (angle, angle, side)	If one side and two angles of a triangle are equal to the corresponding one side and two angles of another triangle, then the two triangles are congruent.	

The order of letters when labelling congruent triangles is very important.

$$\triangle ABC \equiv \triangle DEF$$

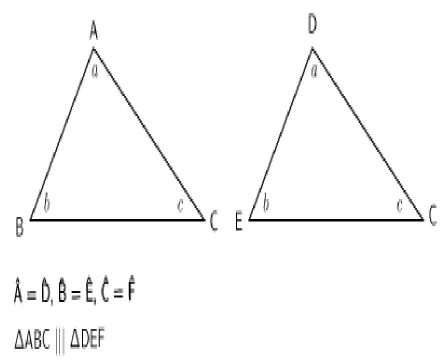
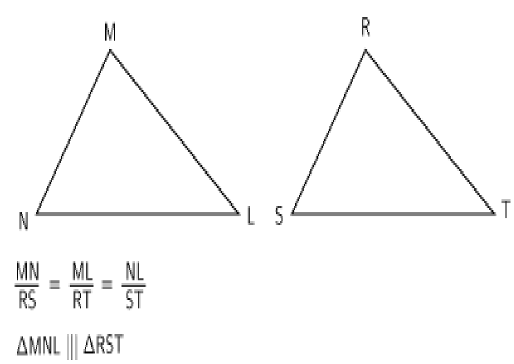
This notation indicates the following properties of the two triangles: $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$, $\hat{C} = \hat{F}$, $AB = DE$, $AC = DF$ and $BC = EF$.

> **NOTE**
>
> You might see $\cong$ used to show that two triangles are congruent. This is the internationally recognised symbol for congruency.

Similarity

Two triangles are similar if one triangle is a scaled version of the other. This means that their corresponding angles are equal in measure and the ratio of their corresponding sides are in proportion. The two triangles have the same shape, but different scales. Congruent triangles are similar triangles, but not all similar triangles are congruent. We use $|||$ to indicate that two triangles are similar.

The following table describes the requirements for similarity:

Rule	Description	Diagram
AAA (angle, angle, angle)	If all three pairs of corresponding angles of two triangles are equal, then the triangles are similar.	
SSS (side, side, side)s	If all three pairs of corresponding sides of two triangles are in proportion, then the triangles are similar.	

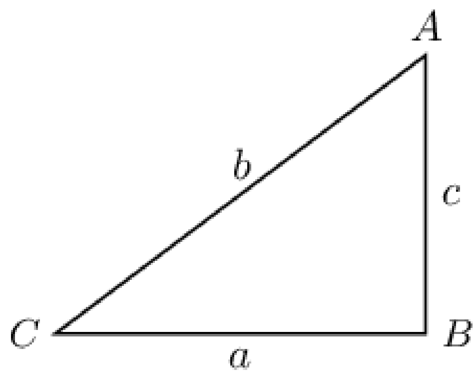
The order of letters for similar triangles is very important. Always label similar triangles in corresponding order. For example,

$\triangle MNL \ ||| \ \triangle RST$ is correct; but
$\triangle MNL \ ||| \ \triangle RTS$ is incorrect.

NOTE

You might see $\sim$ used to show that two triangles are similar. This is the internationally recognised symbol for similarity.

The theorem of Pythagoras



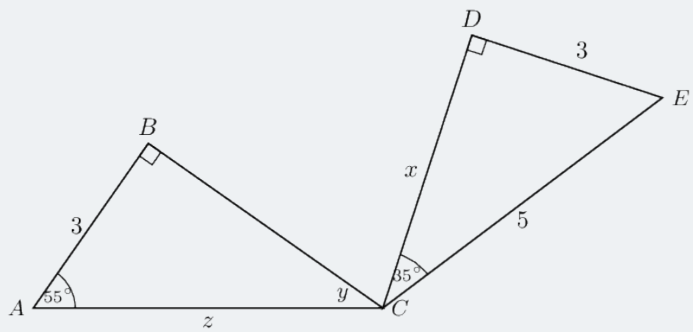
If $\triangle ABC$ is right-angled with $\hat{B} = 90°$, then $b^2 = a^2 + c^2$.

Converse: If $b^2 = a^2 + c^2$, then $\triangle ABC$ is right-angled with $\hat{B} = 90°$.

WORKED EXAMPLE 2: TRIANGLES

Question

Determine if the two triangles are congruent. Use the result to find x, $\hat{y}$ and z.



Solution

Step 1: Examine the information given for both triangles

Step 2 :Determine whether $\triangle CDE \equiv \triangle CBA$

In $\triangle CDE$:

$$\hat{D} + \hat{C} + \hat{E} = 180° \text{(sum of in}\triangle\text{)}$$
$$90° + 35° + \hat{E} = 180°$$
$$\therefore \hat{E} = 55°$$

In $\triangle CDE$ and $\triangle CBA$:

$$D\hat{E}C = B\hat{A}C = 55° \text{(proved)}$$
$$C\hat{D}E = C\hat{B}A = 90° \text{(given)}$$
$$DE = BA = 3 \text{(given)}$$
$$\therefore \triangle CDE = \triangle CBA \text{(AAS)}$$

WORKED EXAMPLE 2: TRIANGLES (CONTINUED)

Step 3: Determine the unknown angles and sides

In $\triangle CDE$:

$$\begin{aligned} CE^2 &= DE^2 + CD^2 \text{ (Pythagoras)} \\ 5^2 &= 3^2 + x^2 \\ x^2 &= 16 \\ \therefore\ x &= 4 \end{aligned}$$

In $\triangle CBA$:

$$\begin{aligned} \hat{B} + \hat{A} + \hat{y} &= 180^\circ \text{ (sum of } \angle\text{s in } \triangle) \\ 90^\circ + 55^\circ + \hat{y} &= 180^\circ \\ \therefore\ \hat{y} &= 35^\circ \\ \triangle CDE &= \triangle CBA \text{ (Proved)} \\ CE &= CA \\ \therefore z &= 5 \end{aligned}$$

6 QUADRILATERALS

DEFINITION

Quadrilateral

A quadrilateral is a closed shape consisting of four straight line segments.

NOTE

The interior angles of a quadrilateral add up to **360**°.

Parallelogram

DEFINITION

Parallelogram

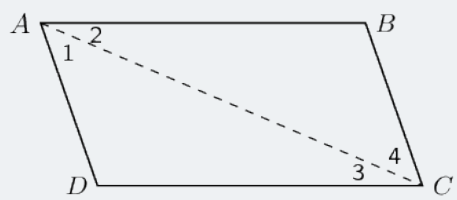
A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

WORKED EXAMPLE 3: PROPERTIES OF A PARALLELOGRAM

Question

$ABCD$ is a parallelogram with $AB \parallel DC$ and $AD \parallel BC$. Show that:

1. AB = DC and AD = BC
2. $\hat{A}$ = $\hat{C}$ and $\hat{B}$ = $\hat{D}$



Solution

Step 1: connect AC to form $\triangle ABC$ and $\triangle CDA$

Redraw the diagram and draw line AC.

Step 2: Use properties of parallel lines to indicate all equal angles on the diagram

On your diagram mark all the equal angles.

Step 3: Prove $\triangle ABC \equiv \triangle CDA$

In $\triangle ABC$ and $\triangle CDA$

$$\hat{A}_2 = \hat{C}_3 \text{ (alt; AB } \parallel \text{ DC)}$$
$$\hat{C}_4 = \hat{A}_1 \text{ (alt; BC } \parallel \text{ AD)}$$

AC (common side)

$\therefore$ $\triangle ABC \equiv \triangle CDA$ (AAS)

$\therefore$ AB = CD and BC = DA

$\therefore$ Opposite sides of a parallelogram have equal length.

WORKED EXAMPLE 3: PROPERTIES OF A PARALLELOGRAM (CONTINUED)

We have already shown that $\hat{A}_2 = \hat{C}_3$ and $\hat{A}_1 = \hat{C}_4$. Therefore,

$$\hat{A} = \hat{A}_1 + \hat{A}_2 = \hat{C}_3 + \hat{C}_4 = \hat{C}$$

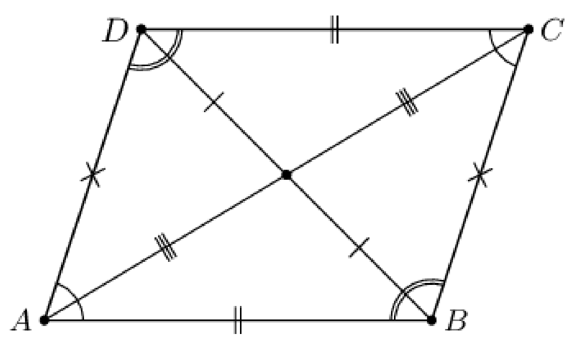
Furthermore,

$$\hat{B} = \hat{D} \quad (\triangle ABC \equiv \triangle CDA)$$

Therefore opposite angles of a parallelogram are equal.

Summary of the properties of a parallelogram:

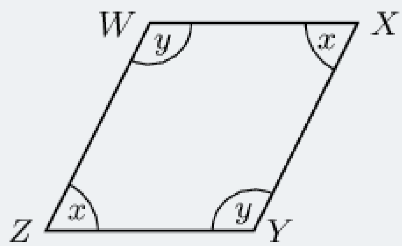
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.



WORKED EXAMPLE 4: PROVING A QUADRILATERAL IS A PARALLELOGRAM

Question

Prove that if both pairs of opposite angles in a quadrilateral are equal, the quadrilateral is a parallelogram.



Solution

Step 1: Find the relationship between $\hat{x}$ and $\hat{y}$

In $WXYZ$:

$$\hat{W} = \hat{Y} = \hat{y} \quad (given)$$
$$\hat{Z} = \hat{X} = \hat{x} \quad (given)$$
$$\hat{W} + \hat{X} + \hat{Y} + \hat{Z} = 360° \quad \text{(sum of in a quad)}$$
$$\therefore 2\hat{x} + 2\hat{y} = 360°$$
$$\therefore \hat{x} + \hat{y} = 180°$$
$$\hat{W} + \hat{Z} = \hat{x} + \hat{y}$$
$$= 180°$$

But these are co-interior angles between lines WX and ZY. Therefore $WX \parallel ZY$.

Step 2: Find parallel lines

Similarly $\hat{W}$ + $\hat{X}$ = $180°$. These are co-interior angles between lines XY and WZ. Therefore $XY \parallel WZ$.

Both pairs of opposite sides of the quadrilateral are parallel, therefore $WXYZ$ is a parallelogram.

INVESTIGATION

Proving a quadrilateral is a parallelogram

1. Prove that if both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
2. Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
3. Prove that if one pair of opposite sides of a quadrilateral are both equal and parallel, then the quadrilateral is a parallelogram.

A quadrilateral is a parallelogram if:

- Both pairs of opposite sides are parallel
- Both pairs of opposite sides are equal.
- Both pairs of opposite angles are equal.
- The diagonals bisect each other.
- One pair of opposite sides are both equal and parallel.

Rectangle

DEFINITION

Rectangle
A rectangle is a parallelogram that has all four angles equal to $\mathbf{90}°$.

A rectangle has all the properties of a parallelogram:

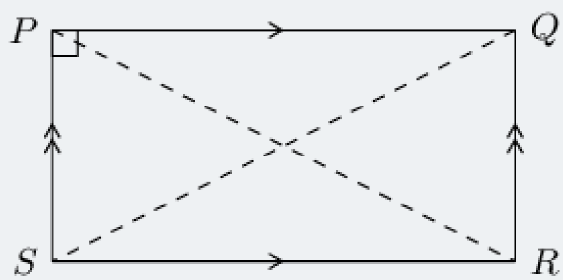
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

It also has the following special property:

WORKED EXAMPLE 5: SPECIAL PROPERTY OF A RECTANGLE

Question

$PQRS$ is a rectangle. Prove that the diagonals are of equal length.



Solution

Step 1: Connect P to R and Q to S to form $\triangle PSR$ and $\triangle QRS$

Step 2: Use the definition of a rectangle to fill in on the diagram all equal angles and sides

Step 3: Prove $\triangle PSR \equiv \triangle QRS$

In $\triangle PSR$ and $\triangle QRS$:

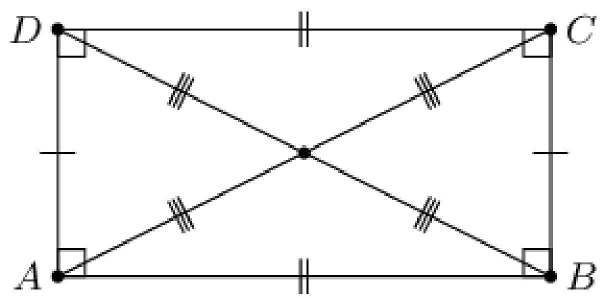
$$PS = QR \quad \text{(opp side of rectangle)}$$
$$SR \quad \text{(common side)}$$
$$P\hat{S}R = Q\hat{R}S = 90^\circ \quad (\angle\text{s of rectangle})$$
$$\therefore \triangle PSR \equiv \triangle QRS \quad \text{(RHS)}$$

Therefore $PR = QS$

The diagonals of a rectangle are of equal length.

Summary of the properties of a rectangle:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are of equal length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- Diagonals are equal in length.
- All interior angles are equal to **90**°.



Rhombus

DEFINITION

Rhombus

A rhombus is a parallelogram with all four sides of equal length.

A rhombus has all the properties of a parallelogram:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

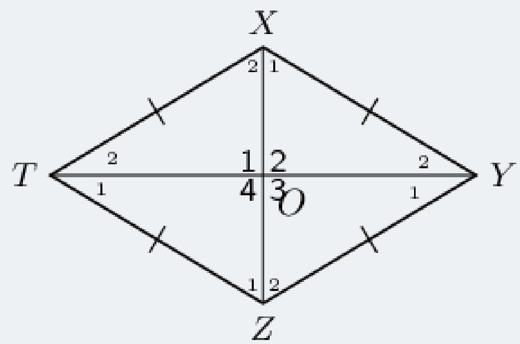
It also has two special properties:

WORKED EXAMPLE 6: SPECIAL PROPERTIES OF A RHOMBUS

Question

$XYZT$ is a rhombus. Prove that:

1. the diagonals bisect each other perpendicularly;
2. the diagonals bisect the interior angles.



Solutions

Step 1: Use the definition of a rhombus to fill in on the diagram all equal angles and sides

Step 2: Prove $\triangle XTO \equiv \triangle ZTO$

$$XT = ZT \quad \text{(sides of rhombus)}$$
$$TO \quad \text{(common side)}$$
$$XO = OZ \quad \text{(diags of rhombus)}$$
$$\therefore \triangle XTO \equiv \triangle ZTO \quad \text{(SSS)}$$
$$\therefore \quad \hat{O}_1 = \hat{O}_4$$
$$\text{But } \hat{O}_1 + \hat{O}_4 = 180^\circ \quad (\angle\text{s on a str line})$$
$$\therefore \quad \hat{O}_1 = \hat{O}_4 = 90^\circ$$

We can further conclude that $\hat{O}_1$ =$\hat{O}_2$ = $\hat{O}_3$ = $\hat{O}_4$ = 90°

Therefore the diagonals bisect each other perpendicularly.

Step 3: Use properties of congruent triangles to prove diagonals bisect interior angles

$$\hat{X}_2 = \hat{Z}_1 \quad (\triangle XTO \equiv \triangle ZTO)$$
$$\text{and } \hat{X}_2 = \hat{Z}_2 \quad (\text{alt } \angle\text{s; } XT \parallel YZ)$$

WORKED EXAMPLE 6: SPECIAL PROPERTIES OF A RHOMBUS (CONTINUED)

$$\therefore \quad \hat{Z}_1 = \hat{Z}_2$$

Therefore diagonal XZ bisects $\hat{Z}$. Similarly, we can show that XZ also bisects $\hat{X}$; and that diagonal TY bisects $\hat{T}$ and $\hat{Y}$.

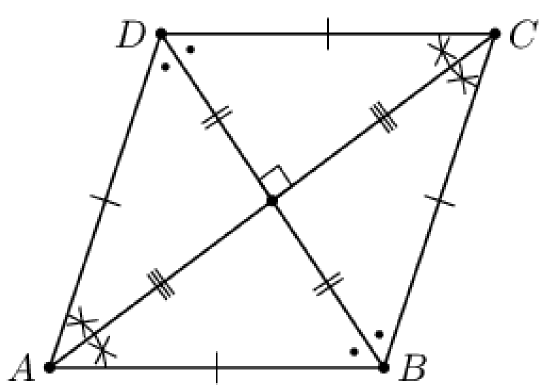
We conclude that the diagonals of a rhombus bisect the interior angles.

To prove a parallelogram is a rhombus, we need to show any one of the following:

- All sides are equal in length.
- Diagonals intersect at right angles.
- Diagonals bisect interior angles.

Summary of the properties of a rhombus:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- All sides are equal in length.
- The diagonals bisect each other at **90**°.
- The diagonals bisect both pairs of opposite angles.



DEFINITION

Square

A square is a rhombus with all four interior angles equal to $\mathbf{90}°$.

OR

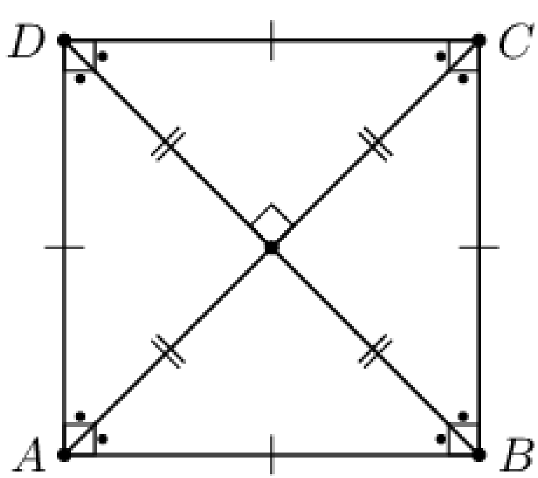
A square is a rectangle with all four sides equal in length.

A square has all the properties of a rhombus:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- All sides are equal in length.
- The diagonals bisect each other at $\mathbf{90}°$.
- The diagonals bisect both pairs of opposite angles.

It also has the following special properties:

- All interior angles equal $\mathbf{90}°$.
- Diagonals are equal in length.
- Diagonals bisect both pairs of interior opposite angles (i.e. all are $\mathbf{45}°$).



To prove a parallelogram is a square, we need to show either one of the following:

- It is a rhombus (all four sides of equal length) with interior angles equal to $\mathbf{90}^{\circ}$.
- It is a rectangle (interior angles equal to $\mathbf{90}^{\circ}$).

Trapezium

DEFINITION

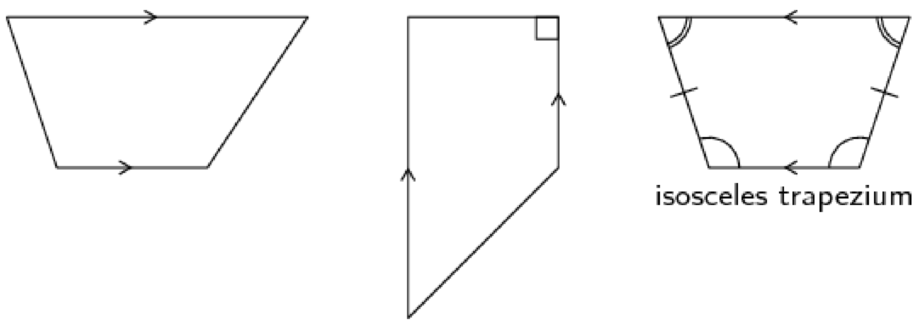
Trapezium

A trapezium is a quadrilateral with one pair of opposite sides parallel.

NOTE

A trapezium is sometimes called a trapezoid.

Some examples of trapeziums are given below:



Kite

DEFINITION

Kite

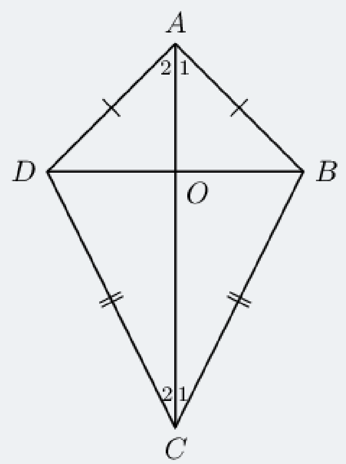
A kite is a quadrilateral with two pairs of adjacent sides equal.

WORKED EXAMPLE 7: SPECIAL PROPERTIES OF A KITE

Question

$ABCD$ is a kite with AD = AB and CD = CB. Prove that:

1. ADC = ABC
2. Diagonal AC bisect $\hat{A}$ and $\hat{C}$



Solution

Step 1: Prove $\triangle ADC \equiv \triangle ABC$:

In $\triangle ADC$ and $\triangle ABC$

$AD = AB$ (given)

$CD = CB$ (given)

AC (common side)

$\therefore \quad \triangle ADC \equiv \triangle ABC \ \ (SSS)$

$\therefore A\hat{D}C = A\hat{B}C$

Therefore one pair of opposite angles are equal in kite $ABCD$.

Step 2: Use properties of congruent triangles to prove AC bisects $\hat{A}$ and $\hat{C}$

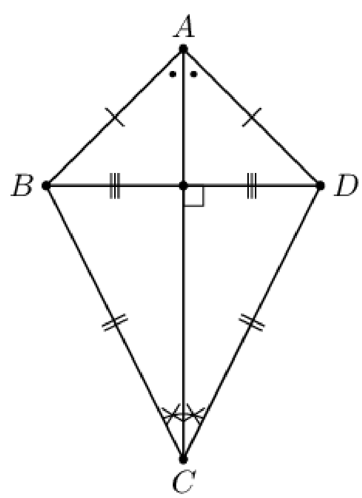
WORKED EXAMPLE 7: SPECIAL PROPERTIES OF A KITE (CONTINUED)

Therefore diagonal AC bisects $\hat{A}$ and $\hat{C}$.

$$\hat{A}_1 = \hat{A}_2 \ \ (\triangle ADC \equiv \triangle ABC)$$
$$and \hat{C}_1 = \hat{C}_2 \ \ (\triangle ADC \equiv \triangle ABC)$$

We conclude that the diagonal between the equal sides of a kite bisects the two interior angles and is an axis of symmetry.

Summary of the properties of a kite:

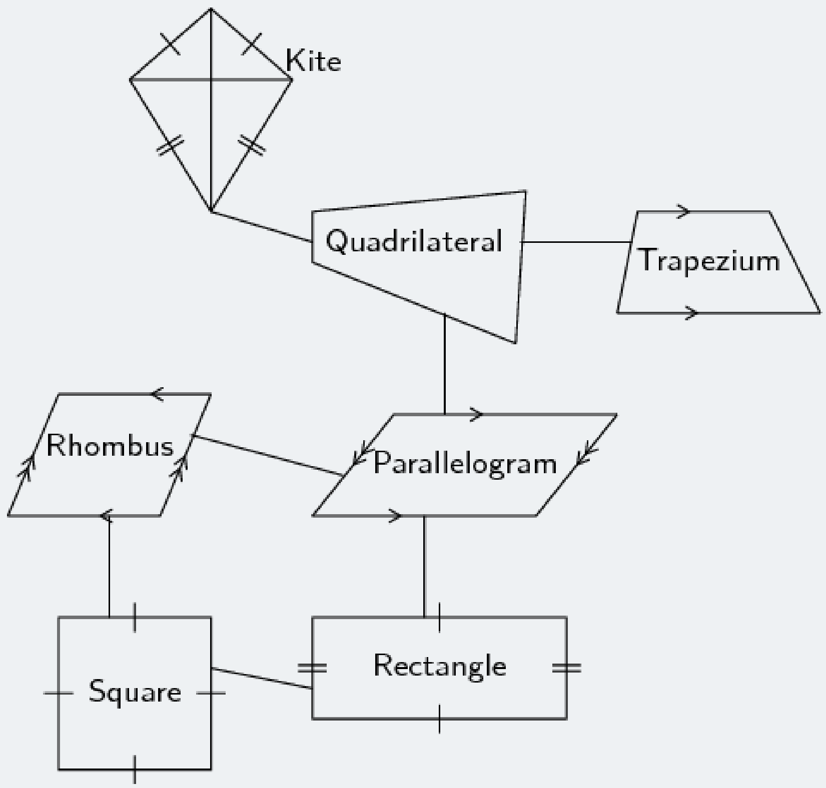


- Diagonal between equal sides bisects the other diagonal.
- One pair of opposite angles are equal (the angles between unequal sides).
- Diagonal between equal sides bisects the interior angles and is an axis of symmetry.
- Diagonals intersect at **90**°.

INVESTIGATION

Relationships between the different quadrilaterals

Heather has drawn the following diagram to illustrate her understanding of the relationships between the different quadrilaterals. The following diagram summarises the different types of special quadrilaterals.

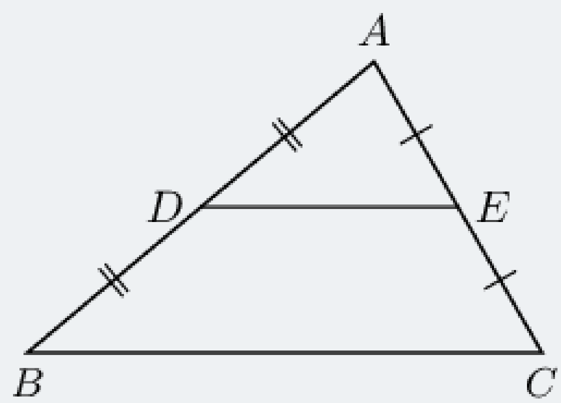


1. Explain her possible reasoning for structuring the diagram as shown.
2. Design your own diagram to show the relationships between the different quadrilaterals and write a short explanation of your design.

7 THE MID-POINT THEOREM

INVESTIGATION

Proving the mid-point theorem

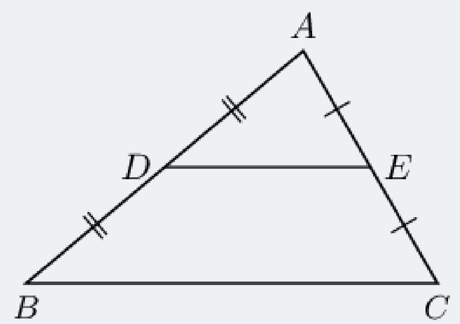


1. Draw a large scalene triangle on a sheet of paper.
2. Name the vertices A, B and C. Find the mid-points (D and E) of two sides and connect them.
3. Cut out $\triangle ABC$ and cut along line DE.
4. Place $\triangle ADE$ on quadrilateral $BDEC$ with vertex E on vertex C. Write down your observations.
5. Shift $\triangle ADE$ to place vertex D on vertex B. Write down your observations.
6. What do you notice about the lengths DE and BC?
7. Make a conjecture regarding the line joining the mid-point of two sides of a triangle.

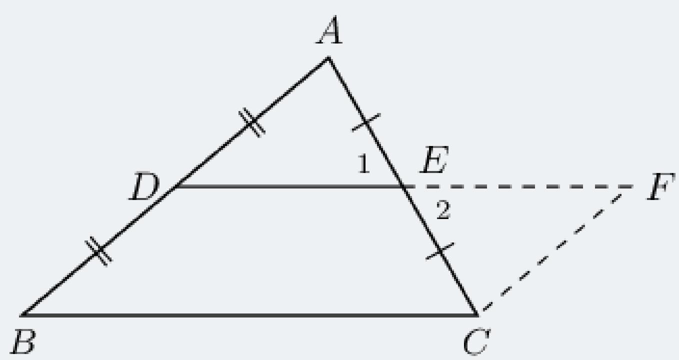
WORKED EXAMPLE 8: MID-POINT THEOREM

Question

Prove that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.



Solution **Step 1: Extend DE to F so that DE=EF and join FC**



Step 2: Prove $BCFD$ is a parallelogram

In $\triangle EAD$ and $\triangle ECF$

$$\hat{E}_1 = \hat{E}_2 \text{ (vert opp=)}$$
$$AE = CE \text{ (given)}$$
$$DE = EF \text{ (by construction)}$$
$$\therefore \triangle EAD = \triangle ECF \ (SAS)$$
$$\therefore A\hat{D}E = C\hat{F}E$$

WORKED EXAMPLE 8: MID-POINT THEOREM (CONTINUED)

But these are alternate interior angles, therefore $BD \parallel FC$

$$BD = DA \text{ (given)}$$
$$DA = FC \quad (\triangle EAD \equiv \triangle ECF)$$
$$\therefore \quad BD = FC$$

$\therefore$ BCFD is a parallelogram

Therefore $DE \parallel BC$

We conclude that the line joining the two mid-points of two sides of a triangle is parallel to the third side.

Step 3: Use properties of parallelogram $BCFD$ to prove that DE = $\frac{1}{2}BC$

$$DF = BC \quad \text{(opp sides of } \parallel \text{ m)}$$
$$\text{and} DF = 2DE \quad \text{(by construction)}$$
$$\therefore \quad 2DE = BC$$
$$\therefore \quad DE = \frac{1}{2}BC$$

We conclude that the line joining the mid-point of two sides of a triangle is equal to half the length of the third side.

Converse

The converse of this theorem states: If a line is drawn through the mid-point of a side of a triangle parallel to the second side, it will bisect the third side.

8 CHAPTER SUMMARY

- A quadrilateral is a closed shape consisting of four straight line segments.
- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
  - Both pairs of opposite sides are equal in length.
  - Both pairs of opposite angles are equal.
  - Both diagonals bisect each other.
- A rectangle is a parallelogram that has all four angles equal to $\mathbf{90}°$.
  - Both pairs of opposite sides are parallel.
  - Both pairs of opposite sides are equal in length.
  - The diagonals bisect each other.
  - The diagonals are equal in length.
  - All interior angles are equal to $\mathbf{90}°$.
- A rhombus is a parallelogram that has all four sides equal in length.
  - Both pairs of opposite sides are parallel.
  - All sides are equal in length.
  - Both pairs of opposite angles are equal.
  - The diagonals bisect each other at $\mathbf{90}°$.
  - The diagonals of a rhombus bisect both pairs of opposite angles.
- A square is a rhombus that has all four interior angles equal to $\mathbf{90}°$.
  - Both pairs of opposite sides are parallel.
  - The diagonals bisect each other at $\mathbf{90}°$.
  - All interior angles are equal to $\mathbf{90}°$.
  - The diagonals are equal in length.
  - The diagonals bisect both pairs of interior opposite angles (i.e. all are $\mathbf{45}°$)
- A trapezium is a quadrilateral with one pair of opposite sides parallel.
- A kite is a quadrilateral with two pairs of adjacent sides equal.
  - One pair of opposite angles are equal (the angles are between unequal sides).
  - The diagonal between equal sides bisects the other diagonal.

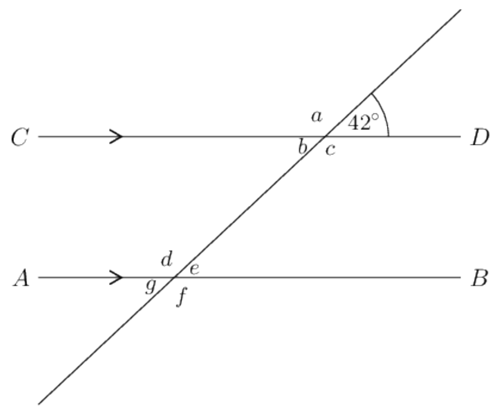
- The diagonal between equal sides bisects the interior angles.
- The diagonals intersect at $90°$.

- The mid-point theorem states that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

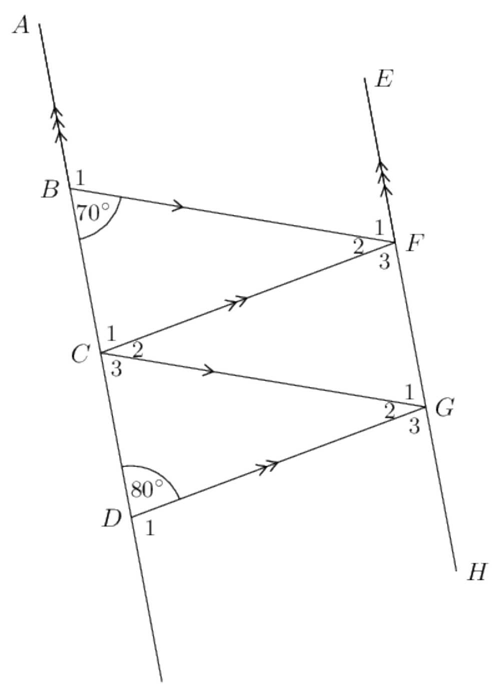
9 EXERCISES

9.1 Exercise 1

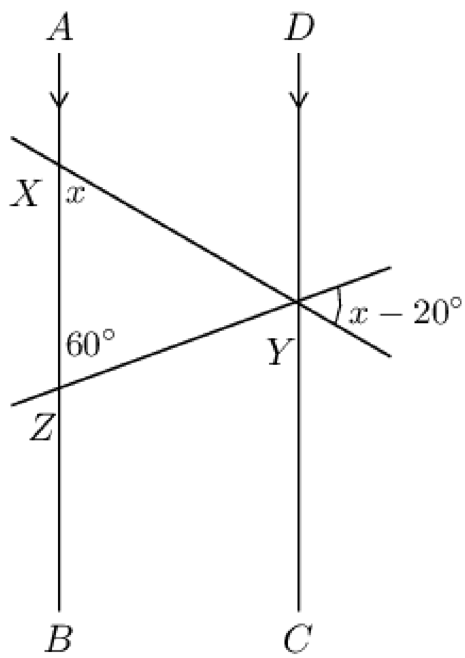
1. Use adjacent, corresponding, co-interior and alternate angles to fill in all the angles labelled with letters in the diagram:



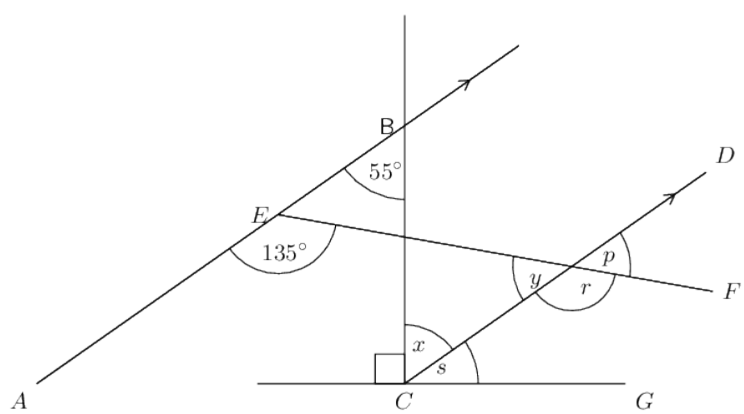
2. Find all the unknown angles in the figure:



3. Find the value of x in the figure:

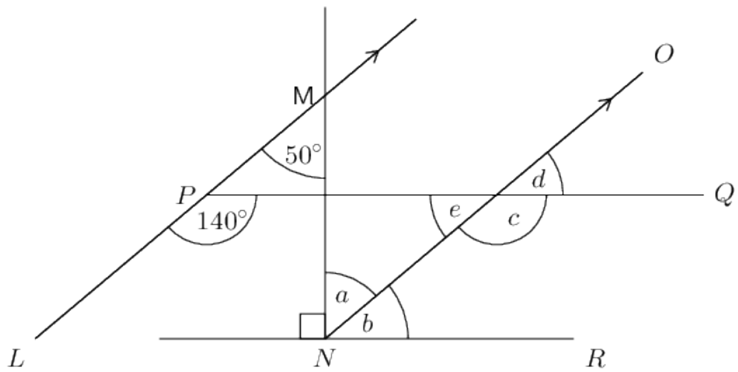


4. Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.



4.1 Based on the results for the angles above, is $EF \parallel CG$

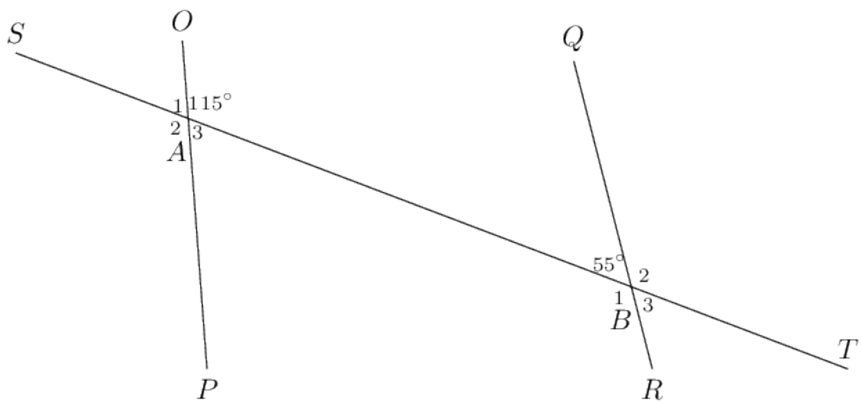
5. Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.



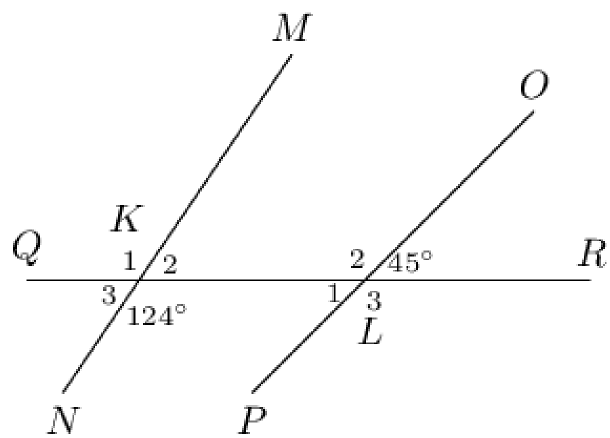
5.1 Based on the results for the angles above, is $PQ \parallel NR$

6. Determine whether the pairs of lines in the following figures are parallel:

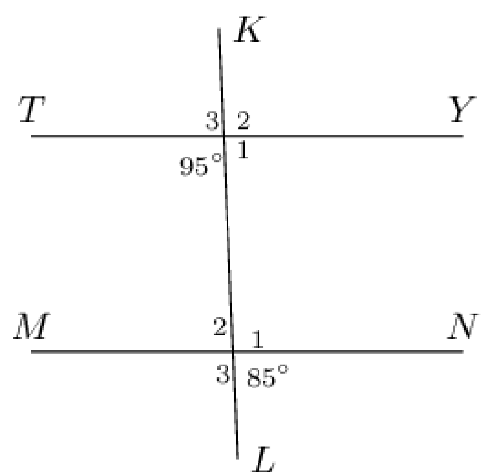
6.1



6.2



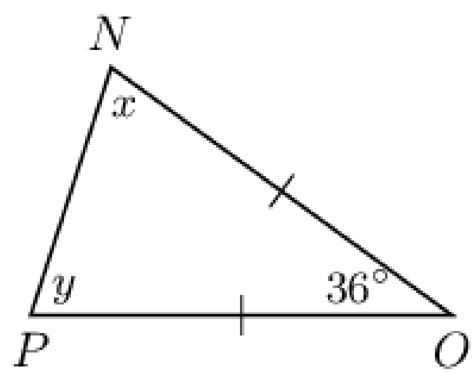
6.3



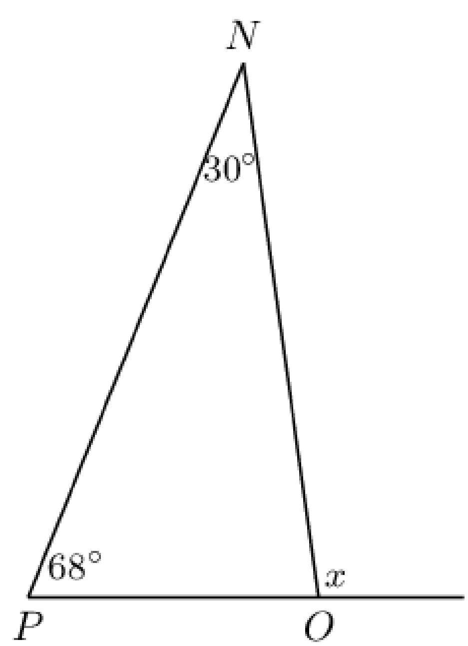
9.2 Exercise 2

1. Calculate the unknown variables in each of the following figures.

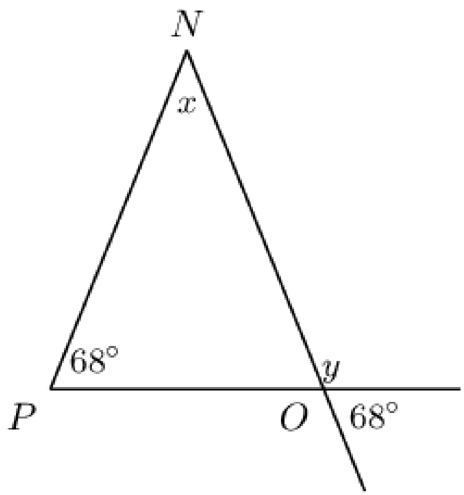
1.1



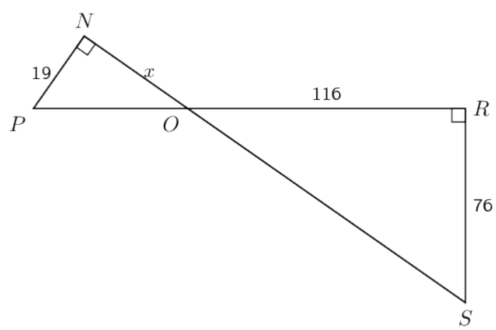
1.2



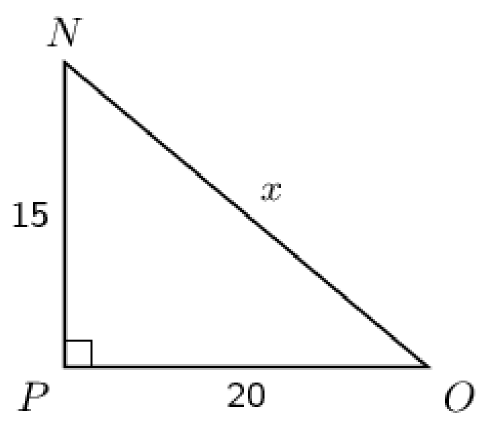
1.3



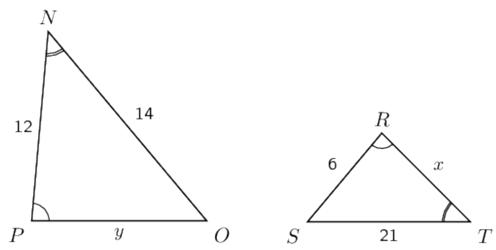
1.4



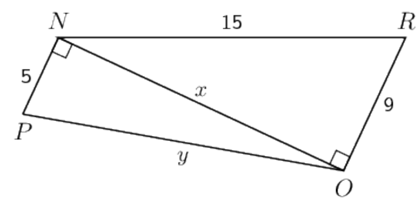
1.5



1.6

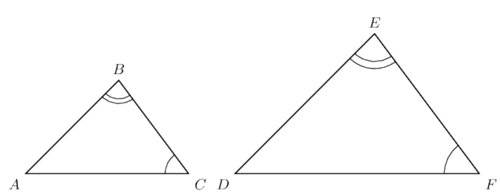


1.7

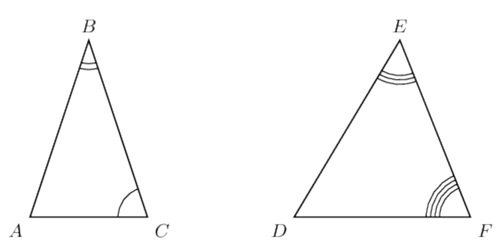


2. Say whether or not the following diagrams show a pair of similar triangles:

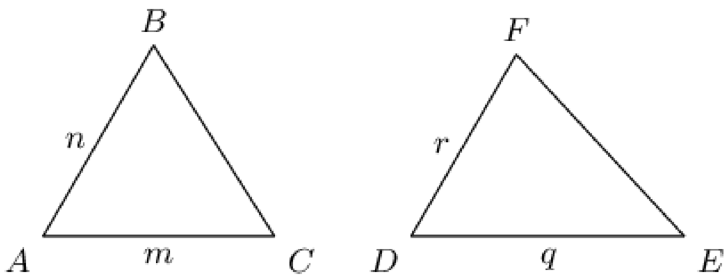
2.1



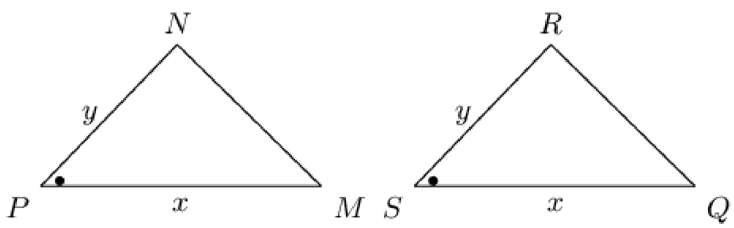
2.2



3. Have a look at the following triangles, which are drawn to scale:
   Are the two triangles congruent?

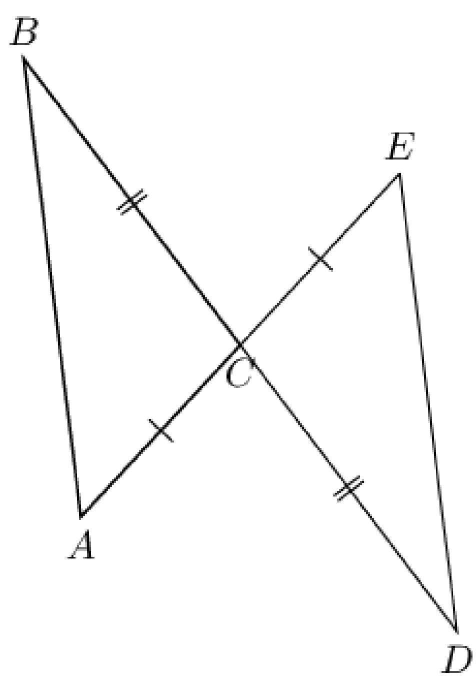


4. Have a look at the following triangles, which are drawn to scale:
   Are the two triangles congruent?

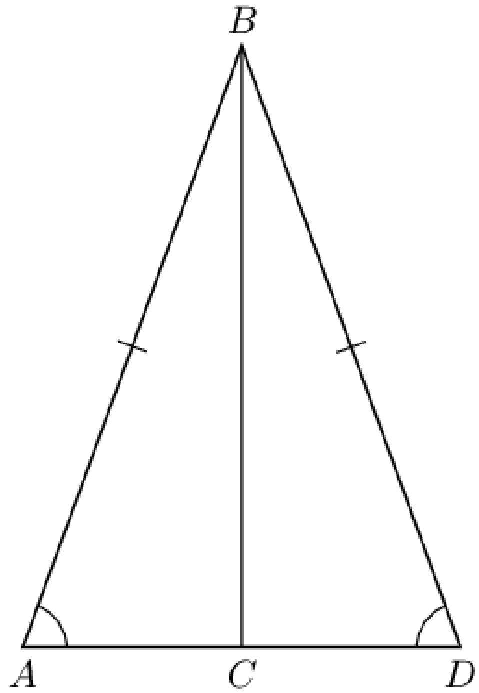


5. State whether the following pairs of triangles are congruent or not.

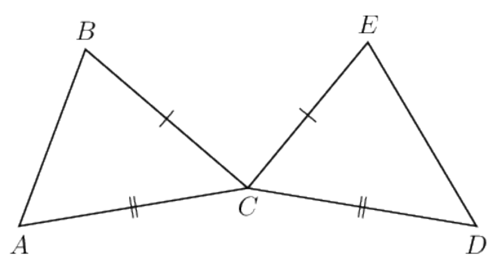
   5.1



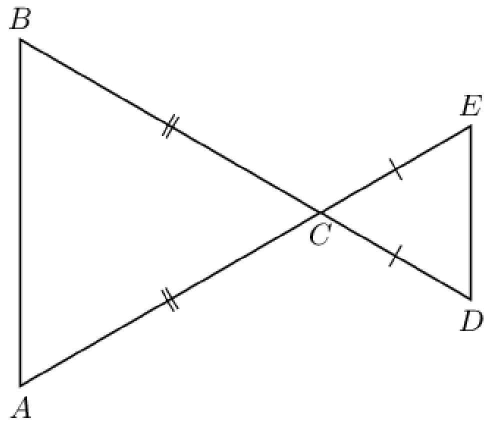
5.2



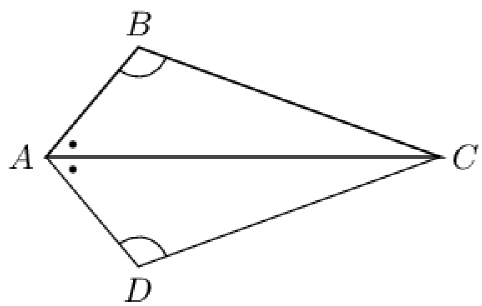
5.3



5.4

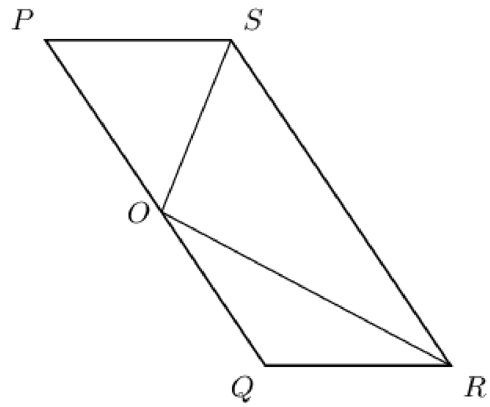


5.5



9.3 Exercise 3

1. $PQRS$ is a parallelogram. $PS = OS$ and $QO = QR$. $S\hat{O}R = 96$ and $Q\hat{O}R = x$

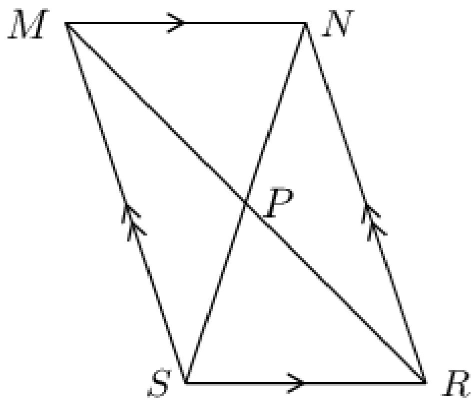


   1.1 Find two other angles equal to x .

   1.2 Write $\hat{P}$ in terms of X .

   1.3 Calculate the value of x .

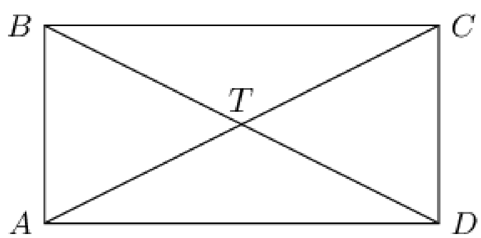
2. Do the diagonals of parallelogram $MNRS$ bisect one another at P ?



Hint: Use congruency.

9.4 Exercise 4

1. $ABCD$ is a quadrilateral. Diagonals AC and BD intersect at T . $AC = BD, AT = TC, DT = TB$.



Determine the following:

1.1 Is $ABCD$ a parallelogram?

1.2 Is $ABCD$ is a rectangle?

9.5 Exercise 5

1. In the diagram below, AC and EF bisect each other at G. E is the midpoint of AD, and F is the midpoint of BC.

   (a) Is $AECF$ a parallelogram?

   (b) Is $ABCD$ a parallelogram?

2. Parallelogram $ABCD$ and EFC are shown below.
   Give the steps to determine that $AB = EF$.

3. In the diagram below $PQRS$ is a parallelogram with $PQ = TQ$.
   Prove $\hat{Q}_1 = \hat{R}$. .

4. Quadrilateral $QRST$ with sides $QR \parallel TS$ and $QT \parallel RS$ is given.
   You are also given that: $\hat{Q} = y$ and $\hat{S} = 34°; Q\hat{T}R = 41°$.

   4.1 Find the value of y.

   4.2 Find the value of x.

5. Quadrilateral $XWVU$ with sides $XW \parallel UV$ and $XU \parallel WV$ is given.
   Also given is $\hat{X} = y$ and $\hat{V} = 36°; X\hat{U}W = 102°$ and $W\hat{U}Vx$.

   5.1 Determine the value of y .

   5.2 Determine the value of x .

10 ANSWERS TO EXERCISES

10.1 Exercise 1

1. $a = 138°$
   $b = 42°$
   $c = 138°$
   $d = 138°$
   $e = 42°$
   $f = 138°$
   $g = 42°$
2. $\hat{B}_1 = 110°$
   $\hat{D}_1 = 100°$
   $\hat{F}_1 = 70°$
   $\hat{G}_3 = 80°$
   $\hat{C}_3 = 70°$
   $\hat{G}_1 = 70°$
   $\hat{G}_2 = 30°$
   $\hat{C}_2 = 30°$
   $\hat{F}_2 = 30°$
   $\hat{F}_3 = 80°$
   $\hat{C}_1 = 80°$
3. $x = 70°$

4.1 $\hat{x} = 55°$

4.2 $\hat{s} = 35°$

4.3 $\hat{r} = 135°$

4.4 $\hat{y} = 45°$

4.5 $\hat{p} = 45°$

4.6 EF is not parallel to CG .

5.1 $\hat{a} = 50°$

5.2 $\hat{b} = 40°$

5.3 $\hat{c} = 140°$

5.4 $\hat{e} = 40°$

5.5 $\hat{d} = 40°$

5.6 PQ is parallel to NR

6.1 No parallel lines

6.2 No parallel lines

6.3 $TY \parallel MN$

10.2 Exercise 2

1.1 $x = y = 72°$

1.2 $x = 98°$

1.3 $x = 44°$
$y = 112°$

1.4 $x = 29$

1.5 $x = 25$

1.6 $x = 18$
$y = 4$

1.7 $x = 12$
$y = 13$

2.1 Similar

2.2 Not similar

3. we cannot say if the two triangles are congruent

4. The two triangles are congruent

5.1 Congruent

5.2 Not congruent

5.3 Not enough information

5.4 Not enough information

5.5 Congruent

10.3 Exercise 3

1.1 $S\hat{R}O$ and $O\hat{R}Q$ are both equal to x

1.2 $\hat{P} = 2x$

1.3 $x = 28°$

2. The diagonals bisect at P .

10.4 Exercise 4

1.1 $ABCD$ is a parallelogram

1.2 $ABCD$ is a rectangle

10.5 Exercise 5

1.1 Yes (Diagonals bisect each other)

1.2 Yes (Two sides parallel and equal)

$AD = EF$

$Q_1 = R$

4.1 $y = 34°$

4.2 $x = 105°$

5.1 $y = 36°$

5.2 $x = 42°$