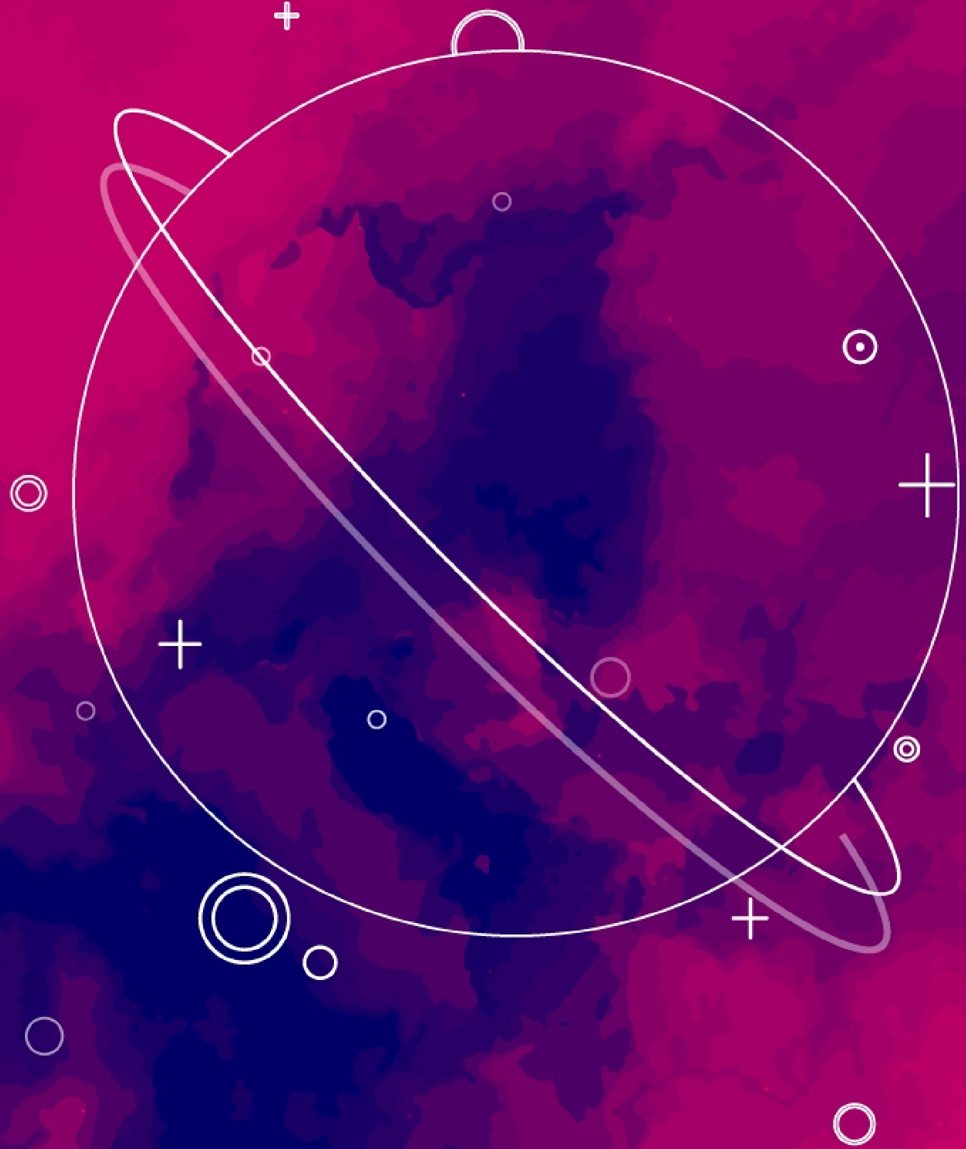


CHAPTER 1

Exponents And Surds

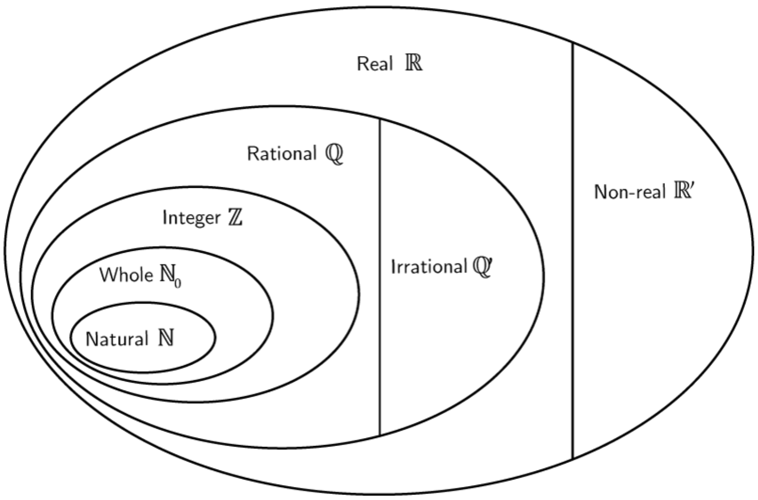
CONTENTS

April 20, 2021

1 REVISION

1.1 The number system

The diagram below shows the structure of the number system: We use the following definitions:



- $\mathbb{N}$: natural numbers are $\{1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{N}_0$: whole numbers are $\{0;\ 1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{Z}$: integers are $\{\ldots;\ -3;\ -2;\ -1;\ 0;\ 1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{Q}$: rational numbers are numbers which can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$ or as a terminating or recurring decimal number.
  Examples: $-\frac{7}{2};\ -2,25;\ 0;\ \sqrt{9};\ 0,\dot{8};\ \frac{23}{1}$
- $\mathbb{Q}'$: Irrational numbers are numbers that cannot be written as a fraction with the numerator and denominator as integers. Irrational numbers also include decimal numbers that neither terminate nor recur.
  Examples: $\sqrt{3};\ \sqrt[5]{2};\ \pi;\ \frac{1+\sqrt{5}}{2};\ 1,27548\ldots$
- $\mathbb{R}$: real numbers include all rational and irrational numbers.
- $\mathbb{R}'$: non-real numbers or imaginary numbers are numbers that are not real.

Examples: $\sqrt{-25};\ \sqrt[4]{-1};\ -\sqrt{-\frac{1}{16}}$

1.2 Laws of exponents

We use exponential notation to show that a number or variable is multiplied by itself a certain number of times. The exponent, also called the index or power, indicates the number of times the multiplication is repeated.

$$\text{base} \longleftarrow a^{n} \longrightarrow \text{exponent/index}$$

$$a^n = a \times a \times a \times \ldots \times a \quad (n \text{ times}) \qquad (a \in \mathbb{R}, n \in \mathbb{N})$$

Examples:

1. $2 \times 2 \times 2 \times 2 = 2^4$
2. $0,71 \times 0,71 \times 0,71 = (0,71)^3$
3. $(501)^2 = 501 \times 501$
4. $k^6 = k \times k \times k \times k \times k \times k$

For x^2, we say x is squared and for y^3 we say that y is cubed. In the last example we have k^6 ; we say that k is raised to the sixth power.
We also have the following definitions for exponents. It is important to remember that we always write the final answer with a positive exponent.

- $a^0 = 1$ (a $\neq 0$ because 0^0 is undefined)
- a$^{-n} = \frac{1}{a^n}$(a $\neq 0$ because $\frac{1}{0}$ is undefined)

Examples:

1. $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$
2. $(-36)^0 x = (1)x = x$
3. $\dfrac{7p^{-1}}{q^3 t^{-2}} = \dfrac{7t^2}{pq^3}$

MathU Teaching® Emporium

We use the following laws for working with exponents:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $(a^m)^n = a^{mn}$

where $a > 0$, $b > 0$ and $m, n \in \mathbb{Z}$

Example : WORKED EXAMPLE 1: LAWS OF EXPONENTS

QUESTION

Simplify the following:

1. $5(m^{2t})^p \times 2(m^{3p})^t$
2. $\frac{8k^3x^2}{(xk)^2}$
3. $\frac{2^2 \times 3 \times 7^4}{(7 \times 2)^4}$
4. $3(3^b)^a$

SOLUTION

1. $5(m^{2t})^p \times 2(m^{3p})^t = 10m^{2pt+3pt} = 10m^{5pt}$
2. $\frac{8k^3x^2}{(xk)^2} = \frac{8k^3x^2}{x^2k^2} = 8k^{(3-2)}x^{(2-2)} = 8k^1x^0 = 8k$
3. $\frac{2^2 \times 3 \times 7^4}{(7 \times 2)^4} = \frac{2^2 \times 3 \times 7^4}{7^4 \times 2^4} = 2^{(2-4)} \times 3 \times 7^{(4-4)} = 2^{-2} \times 3 = \frac{3}{4}$
4. $3(3^b)^a = 3 \times 3^{ab} = 3^{ab+1}$

Example : WORKED EXAMPLE 2: LAWS OF EXPONENTS

QUESTION

Simplify: $\frac{3^m - 3^{m+1}}{4 \times 3^m - 3^m}$

SOLUTION

Step 1: Simplify to a form that can be factorised

$$\frac{3^m - 3^{m+1}}{4 \times 3^m - 3^m} = \frac{3^m - (3^m \times 3)}{4 \times 3^m - 3^m}$$

Step 2: Take out a common factor

$$= \frac{3^m(1-3)}{3^m(4-1)}$$

Step 3: Divide the common factor and simplify

$$= \frac{1-3}{4-1}$$
$$= -\frac{2}{3}$$

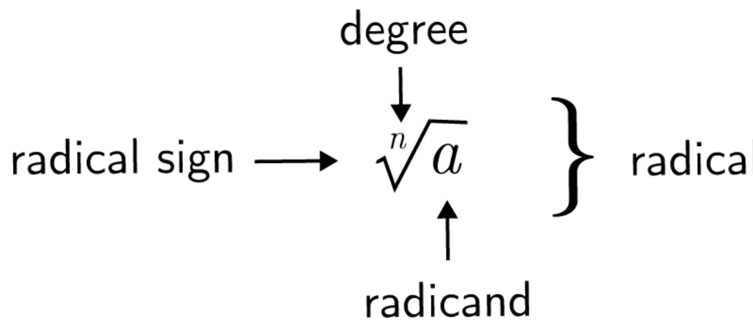
2 RATIONAL EXPONENTS AND SURDS

The laws of exponents can also be extended to include the rational numbers. A rational number is any number that can be written as a fraction with an integer in the numerator and in the denominator. We also have the following definitions for working with rational exponents.

- If $r^n = a$ then $r = \sqrt[n]{a} \quad (n \geq 2)$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{-\frac{1}{n}} = (a^{-1})^{\frac{1}{n}} = \sqrt[n]{\frac{1}{a}}$
- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

Where $a > 0$, $r > 0$ and $m, n \in \mathbb{Z}$, $n \neq 0$.
For $\sqrt{25} = 5$ we say that 5 is the square root of 25 and for, $\sqrt[3]{8} = 2$ we say that 2 is the cube root of 8. For $\sqrt[5]{32} = 2$ we say that 2 is the fifth root of 32.
When dealing with exponents, a root refers to a number that is repeatedly multiplied by itself a certain number of times to get another number. A radical refers to a number written as shown below.

The radical symbol and degree show which root is being determined. The radicand is the number under the radical symbol.



- If n is an even natural number, then the radicand must be positive, otherwise the roots are not real. For example, $\sqrt[4]{16} = 2$ since $2 \times 2 \times 2 \times 2 = 16$, but the roots of $\sqrt[4]{-16}$ are not real since $(-2) \times (-2) \times (-2) \times (-2) \neq -16$.
- If n is an odd natural number, then the radicand can be positive or negative. For example, $\sqrt[3]{27} = 3$ since $3 \times 3 \times 3 = 27$ and we can also determine $\sqrt[3]{-27} = -3$ since $(-3) \times (-3) \times (-3) = -27$.

It is also possible for there to be more than one n^{th} root of a number. For example, $(-2)^2 = 4$ and $2^2 = 4$, so both -2 and 2 are square roots of 4.
A surd is a radical which results in an irrational number. Irrational numbers are numbers that cannot be written as a fraction with the numerator and the denominator as integers. For example, $\sqrt{12}$, $\sqrt[3]{100}$, $\sqrt[5]{25}$ are surds.

Example : WORKED EXAMPLE 3: RATIONAL EXPONENTS

QUESTION Write each of the following as a radical and simplify where possible:

1. $18^{\frac{1}{2}}$
2. $(-125)^{-\frac{1}{3}}$
3. $4^{\frac{3}{2}}$
4. $(-81)^{\frac{1}{2}}$
5. $(0{,}008)^{\frac{1}{3}}$

SOLUTION

1. $18^{\frac{1}{2}} = \sqrt{18}$
2. $(-125)^{-\frac{1}{3}} = \sqrt[3]{(-125)^{-1}} = \sqrt[3]{\frac{1}{-125}} = \sqrt[3]{\frac{1}{(-5)^3}} = -\frac{1}{5}$
3. $4^{\frac{3}{3}} = (2^2)^{\frac{3}{2}} = 2^3 = 8$
4. $(-81)^{\frac{1}{2}} = \sqrt{-81} =$ not real
5. $(0{,}008)^{\frac{1}{3}} = \sqrt[3]{\frac{8}{1\ 000}} = \sqrt[3]{\frac{2^3}{10^3}} = \frac{2}{10} = \frac{1}{5}$

Example : WORKED EXAMPLE 4: RATIONAL EXPONENTS

QUESTION

Simplify without using a calculator:

$$\left(\frac{5}{4^{-1}-9^{-1}}\right)^{\frac{1}{2}}$$

SOLUTION

Step 1: Write the fraction with positive exponents in the denominator

$$\left(\frac{5}{\frac{1}{4}-\frac{1}{9}}\right)^{\frac{1}{2}}$$

Step 2: Simplify the denominator

$$\begin{aligned}
&= \left(\frac{5}{\frac{9-4}{36}}\right)^{\frac{1}{2}} \\
&= \left(\frac{5}{\frac{5}{36}}\right)^{\frac{1}{2}} \\
&= \left(5 \div \frac{5}{36}\right)^{\frac{1}{2}} \\
&= \left(5 \times \frac{36}{5}\right)^{\frac{1}{2}} \\
&= (36)^{\frac{1}{2}}
\end{aligned}$$

Step 3: Take the square root

$$\begin{aligned}
&= \sqrt{36} \\
&= 6
\end{aligned}$$

2.1 Simplification of surds

We have seen in previous examples and exercises that rational exponents are closely related to surds. It is often useful to write a surd in exponential notation as it allows us to use the exponential laws.

The additional laws listed below make simplifying surds easier:

- $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
- $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
- $(\sqrt[n]{a})^m = a^{\frac{m}{n}}$

Example : WORKED EXAMPLE 5: SIMPLIFYING SURDS

QUESTION

Show that:

1. $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

SOLUTION

1.

$$\begin{aligned}\sqrt[n]{a} \times \sqrt[n]{b} &= a^{\frac{1}{n}} \times b^{\frac{1}{n}} \\ &= (ab)^{\frac{1}{n}} \\ &= \sqrt[n]{ab}\end{aligned}$$

2.

$$\begin{aligned}\sqrt[n]{\frac{a}{b}} &= \left(\frac{a}{b}\right)^{\frac{1}{n}} \\ &= \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \\ &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}}\end{aligned}$$

Examples:

1. $\sqrt{2} \times \sqrt{32} = \sqrt{2 \times 32} = \sqrt{64} = 8$
2. $\frac{\sqrt[3]{24}}{\sqrt[3]{3}} = \sqrt[3]{\frac{24}{3}} = \sqrt[3]{8} = 2$
3. $\sqrt{\sqrt{81}} = \sqrt[4]{81} = \sqrt[4]{3^4} = 3$

2.2 Like and unlike surds

Two surds $\sqrt[m]{a}$ and $\sqrt[n]{b}$ are like surds if $m = n$, otherwise they are called unlike surds. For example, $\sqrt{\frac{1}{3}}$ and $-\sqrt{61}$ are like surds because $m = n = 2$. Examples of unlike surds are $\sqrt[3]{5}$ and $\sqrt[5]{7y^3}$ since $m \neq n$.

2.3 Simplest surd form

We can sometimes simplify surds by writing the radicand as a product of factors that can be further simplified using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

Example : WORKED EXAMPLE 6: SIMPLEST SURD FORM

QUESTION

Write the following in simplest surd form: $\sqrt{50}$.

SOLUTION

Step 1: Write the radicand as a product of prime factors

$$\begin{aligned} \sqrt{50} &= \sqrt{5 \times 5 \times 2} \\ &= \sqrt{5^2 \times 2} \end{aligned}$$

Step 2: Simplify using: $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned} &= \sqrt{5^2} \times \sqrt{2} \\ &= 5 \times \sqrt{2} \\ &= 5\sqrt{2} \end{aligned}$$

Sometimes a surd cannot be simplified. For example, $\sqrt{6}$, $\sqrt[3]{30}$ and $\sqrt[4]{42}$ are already in their simplest form.

Example : WORKED EXAMPLE 7: SIMPLEST SURD FORM

QUESTION

Write the following in simplest surd form: $\sqrt[3]{54}$

SOLUTION

Step 1: Write the radicand as a product of prime factors

$$\begin{aligned}\sqrt[3]{54} &= \sqrt[3]{3 \times 3 \times 3 \times 2} \\ &= \sqrt[3]{3^3 \times 2}\end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned}&= \sqrt[3]{3^3} \times \sqrt[3]{2} \\ &= 3 \times \sqrt[3]{2} \\ &= 3\sqrt[3]{2}\end{aligned}$$

Example : WORKED EXAMPLE 8: SIMPLEST SURD FORM

QUESTION

Write the following in simplest surd form: $\sqrt[3]{54}$.

SOLUTION

Step 1: Write the radicand as a product of prime factors

$$\begin{aligned}\sqrt[3]{54} &= \sqrt[3]{3 \times 3 \times 3 \times 2} \\ &= \sqrt[3]{3^3 \times 2}\end{aligned}$$

Step 2:Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned}&= \sqrt[3]{3^3} \times \sqrt[3]{2} \\ &= 3 \times \sqrt[3]{2} \\ &= 3\sqrt[3]{2}\end{aligned}$$

Example : WORKED EXAMPLE 8: SIMPLEST SURD FORM

QUESTION

Simplify: $\sqrt{147} + \sqrt{108}$

SOLUTION

Step 1: Write the radicands as a product of prime factors

$$\begin{aligned}\sqrt{147} + \sqrt{108} &= \sqrt{49 \times 3} + \sqrt{36 \times 3} \\ &= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3}\end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned}&= \left(\sqrt{7^2} \times \sqrt{3}\right) + \left(\sqrt{6^2} \times \sqrt{3}\right) \\ &= \left(7 \times \sqrt{3}\right) + \left(6 \times \sqrt{3}\right) \\ &= 7\sqrt{3} + 6\sqrt{3}\end{aligned}$$

Step 3: Simplify and write the final answer

$$13\sqrt{3}$$

Example : WORKED EXAMPLE 9: SIMPLEST SURD FORM

QUESTION

Simplify: $(\sqrt{20} - \sqrt{5})^2$

SOLUTION

Step 1: Factorise the radicands were possible

$$(\sqrt{20} - \sqrt{5})^2 = (\sqrt{4 \times 5} - \sqrt{5})^2$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned}&= \left(\sqrt{4} \times \sqrt{5} - \sqrt{5}\right)^2 \\ &= \left(2 \times \sqrt{5} - \sqrt{5}\right)^2 \\ &= \left(2\sqrt{5} - \sqrt{5}\right)^2\end{aligned}$$

Step 3: Simplify and write the final answer

$$\begin{aligned}&= \left(\sqrt{5}\right)^2 \\ &= 5\end{aligned}$$

Example : WORKED EXAMPLE 10: SIMPLEST SURD FORM WITH FRACTIONS

QUESTION

Write in simplest surd form: $\sqrt{75} \times \sqrt[3]{(48)^{-1}}$

SOLUTION

Step 1: Factorise the radicands were possible

$$\begin{aligned}\sqrt{75} \times \sqrt[3]{(48)^{-1}} &= \sqrt{25 \times 3} \times \sqrt[3]{\frac{1}{48}} \\ &= \sqrt{25 \times 3} \times \frac{1}{\sqrt[3]{8 \times 6}}\end{aligned}$$

Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned}&= \sqrt{25} \times \sqrt{3} \times \frac{1}{\sqrt[3]{8} \times \sqrt[3]{6}} \\ &= 5 \times \sqrt{3} \times \frac{1}{2 \times \sqrt[3]{6}}\end{aligned}$$

Step 3: Simplify and write the final answer

$$\begin{aligned}&= 5\sqrt{3} \times \frac{1}{2\sqrt[3]{6}} \\ &= \frac{5\sqrt{3}}{2\sqrt[3]{6}}\end{aligned}$$

2.4 Rationalising denominators

It is often easier to work with fractions that have rational denominators instead of surd denominators. By rationalising the denominator, we convert a fraction with a surd in the denominator to a fraction that has a rational denominator.

Example : WORKED EXAMPLE 11: RATIONALISING THE DENOMINATOR

QUESTION

Rationalise the denominator:

$$\frac{5x-16}{\sqrt{x}}$$

SOLUTION

Step 1: Multiply the fraction by $\frac{\sqrt{x}}{\sqrt{x}}$ Notice that, $\frac{\sqrt{x}}{\sqrt{x}} = 1$ so the value of the fraction has not been changed.

$$\frac{5x-16}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}(5x-16)}{\sqrt{x}\times\sqrt{x}}$$

Step 2: Simplify the denominator

$$= \frac{\sqrt{x}(5x-16)}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}(5x-16)}{x}$$

The term in the denominator has changed from a surd to a rational number. Expressing the surd in the numerator is the preferred way of writing expressions.

Example : WORKED EXAMPLE 12: RATIONALISING THE DENOMINATOR

QUESTION

Write the following with a rational denominator:

$$\frac{y-25}{\sqrt{y}+5}$$

SOLUTION

Step 1: Multiply the fraction by $\frac{\sqrt{y}-5}{\sqrt{y}-5}$

To eliminate the surd from the denominator, we must multiply the fraction by an expression that will result in a difference of two squares in the denominator.

$$\frac{y-25}{\sqrt{y}+5} \times \frac{\sqrt{y}-5}{\sqrt{y}-5}$$

Step 2: Simplify the denominator

$$\begin{aligned} &= \frac{(y-25)(\sqrt{y}-5)}{(\sqrt{y}+5)(\sqrt{y}-5)} \\ &= \frac{(y-25)(\sqrt{y}-5)}{(\sqrt{y})^2-25} \\ &= \frac{(y-25)(\sqrt{y}-5)}{y-25} \\ &= \sqrt{y}-5 \end{aligned}$$

3 SOLVING SURD EQUATIONS

Example : WORKED EXAMPLE 13: SURD EQUATIONS

QUESTION

Solve for x: $5\sqrt[3]{x^4} = 405$

SOLUTION

Step 1: Write in exponential notation

$$5\left(x^4\right)^{\frac{1}{3}} = 405$$
$$5x^{\frac{4}{3}} = 405$$

Step 2: Divide both sides of the equation by 5 and simplify

$$\frac{5x^{\frac{4}{3}}}{5} = \frac{405}{5}$$
$$x^{\frac{4}{3}} = 81$$
$$x^{\frac{4}{3}} = 3^4$$

Step 3: Simplify the exponents

$$\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} = \left(3^4\right)^{\frac{3}{4}}$$
$$x = 3^3$$
$$x = 27$$

Step 4: Check the solution by substituting the answer back into the original equation

$$\begin{aligned} \text{LHS} &= 5\sqrt[3]{x^4} \\ &= 5(27)^{\frac{4}{3}} \\ &= 5(3^3)^{\frac{4}{3}} \\ &= 5(3^4) \\ &= 405 \\ &= \text{RHS} \end{aligned}$$

Example : WORKED EXAMPLE 14: SURD EQUATIONS

QUESTION

Solve for z: $z - 4\sqrt{z} + 3 = 0$

SOLUTION

Step 1: Factorise

$$\begin{aligned} z - 4\sqrt{z} + 3 &= 0 \\ z - 4z^{\frac{1}{2}} + 3 &= 0 \\ (z^{\frac{1}{2}} - 3)(z^{\frac{1}{2}} - 1) &= 0 \end{aligned}$$

Step 2: Solve for both factors The zero law states: if $a \times b = 0$, then $a = 0$ or $b = 0$.

$$\therefore (z^{\frac{1}{2}} - 3) = 0 \text{ or } (z^{\frac{1}{2}} - 1) = 0$$

Therefore

$$\begin{aligned} z^{\frac{1}{2}} - 3 &= 0 \\ z^{\frac{1}{2}} &= 3 \\ \left(z^{\frac{1}{2}}\right)^2 &= 3^2 \\ z &= 9 \end{aligned}$$

or

$$\begin{aligned} z^{\frac{1}{2}} - 1 &= 0 \\ z^{\frac{1}{2}} &= 1 \\ \left(z^{\frac{1}{2}}\right)^2 &= 1^2 \\ z &= 1 \end{aligned}$$

Step 3: Check the solution by substituting both answers back into the original equation

If $z = 9$:

$$\begin{aligned} \text{LHS} &= z - 4\sqrt{z} + 3 \\ &= 9 - 4\sqrt{9} + 3 \\ &= 12 - 12 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Example : WORKED EXAMPLE 14: SURD EQUATIONS (continued)

If $z = 1$:

$$\begin{aligned} \text{LHS} &= z - 4\sqrt{z} + 3 \\ &= 1 - 4\sqrt{1} + 3 \\ &= 4 - 4 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Step 4: Write the final answer

The solution to $z - 4\sqrt{z} + 3 = 0$ is $z = 9$ or $z = 1$.

Example : WORKED EXAMPLE 15: SURD EQUATIONS

QUESTION

Solve for p: $\sqrt{p-2}-3=0$

SOLUTION

Step 1: Write the equation with only the square root on the left hand side

Use the additive inverse to get all other terms on the right hand side and only the square root on the left hand side.

$$\sqrt{p-2}=3$$

Step 2: Square both sides of the equation

$$\begin{aligned}\left(\sqrt{p-2}\right)^2 &= 3^2\\ p-2 &= 9\\ p &= 11\end{aligned}$$

Step 3: Check the solution by substituting the answer back into the original equation

If $p=11$:

$$\begin{aligned}\text{LHS} &= \sqrt{p-2}-3\\ &= \sqrt{11-2}-3\\ &= \sqrt{9}-3\\ &= 3-3\\ &= 0\\ &= \text{RHS}\end{aligned}$$

Step 4: Write the final answer

The solution to $\sqrt{p-2}-3=0$ is $p=11$.

4 APPLICATIONS OF EXPONENTIALS

There are many real world applications that require exponents. For example, exponentials are used to determine population growth and they are also used in finance to calculate different types of interest.

Example : WORKED EXAMPLE 16: APPLICATIONS OF EXPONENTIALS

QUESTION

A type of bacteria has a very high exponential growth rate at 80% every hour. If there are 10 bacteria, determine how many there will be in five hours, in one day and in one week?

SOLUTION

Step 1: Exponential formula

$$\text{final population} = \text{initial population} \times (1 + \text{growth percentage})^{\text{time period in months}}$$

Therefore, in this case:

$$\text{final population} = 10\,(1{,}8)^n$$

where n= number of hours.

Step 2: In 5 hours

final population $=10\,(1{,}8)^5 \approx 189$

Step 3: In 1 day = 24 hours final population

$=10\,(1{,}8)^{24} \approx 13\ 382\ 588$

Step 4: In 1 week = 168 hours final population

$=10\,(1{,}8)^{168} \approx 7{,}687 \times 10^{43}$

Note this answer is also given in scientific notation as it is a very big number.

Example : WORKED EXAMPLE 17: APPLICATIONS OF EXPONENTIALS

QUESTION

A species of extremely rare deep water fish has a very long lifespan and rarely has offspring. If there are a total of 821 of this type of fish and their growth rate is 2% each month, how many will there be in half of a year? What will the population be in ten years and in one hundred years?

SOLUTION

Step 1: Exponential formula

$$\text{final population} = \text{initial population} \times (1 + \text{growth percentage})^{\text{time period in months}}$$

Therefore, in this case:

$$\text{final population} = 821(1,02)^n$$

where n= number of hours.

Step 2: In half a year = 6 months

final population $= 821(1,02)^6 \approx 925$

Step 3: In 10 years = 120 months

final population $= 821(1,02)^{120} \approx 8\,838$

Step 4: In 100 years = 1 200 months

final population $= 821(1,02)^{1200} \approx 1,716 \times 10^{13}$

Note this answer is also given in scientific notation as it is a very big number.

5 SUMMARY

- The number system:
  - $\mathbb{N}$: natural numbers are $\{1;\ 2;\ 3;\ \ldots\}$
  - $\mathbb{N}_0$: whole numbers are $\{0;\ 1;\ 2;\ 3;\ \ldots\}$
  - $\mathbb{Z}$: integers are $\{\ldots;\ -3;\ -2;\ -1;\ 0;\ 1;\ 2;\ 3;\ \ldots\}$
  - $\mathbb{Q}$: rational numbers are numbers which can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$ or as a terminating or recurring decimal number.
    Examples: $-\frac{7}{2};\ -2,25;\ 0;\ \sqrt{9};\ 0,\dot{8};\ \frac{23}{1}$
  - $\mathbb{Q}'$: Irrational numbers are numbers that cannot be written as a fraction with the numerator and denominator as integers. Irrational numbers also include decimal numbers that neither terminate nor recur.
    Examples: $\sqrt{3};\ \sqrt[5]{2};\ \pi;\ \frac{1+\sqrt{5}}{2};\ 1,27548\ldots$
  - $\mathbb{R}$: real numbers include all rational and irrational numbers.
  - $\mathbb{R}'$: non-real numbers or imaginary numbers are numbers that are not real.
- Definitions:
  - $a^n = a \times a \times a \times \cdots \times a\ (n \text{ times})\ (a \in \mathbb{R}, n \in \mathbb{N})$
  - $a^0 = 1$ ($a \neq 0$ because 0^0 is undefined)
  - $a^{-n} = \frac{1}{a^n}$ ($a \neq 0$ because $\dfrac{1}{0}$ is undefined)
- Laws of exponents:
  - $a^m \times a^n = a^{m+n}$
  - $\dfrac{a^m}{a^n} = a^{m-n}$
  - $(ab)^n = a^n b^n$
  - $\left(\frac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
  - $(a^m)^n = a^{mn}$ where $a > 0$, $b > 0$ and $m, n \in \mathbb{Z}$.
- Rational exponents and surds:
  - if $r^n = a$, then $r = \sqrt[n]{a} \quad (n \geq 2)$
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{-\frac{1}{n}} = (a^{-1})^{\frac{1}{n}} = \sqrt[n]{\dfrac{1}{a}}$
  - $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$ where $a > 0$, $r > 0$ and $m, n \in \mathbb{Z}$.

- Simplification of surds:
  - $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$
  - $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
  - $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

6 EXERCISES

6.1 Exercise 1

1. Use the list of words below to describe each of the following numbers (in some cases multiple words will be applicable):

   - Natural $(\mathbb{N})$
   - Whole $(\mathbb{N}_0)$
   - Integer $(\mathbb{Z})$
   - Rational $(\mathbb{Q})$
   - Irrational $(\mathbb{Q}')$
   - Real $(\mathbb{R})$
   - Non-real $(\mathbb{R}')$

   1.1 $\sqrt{7}$

   1.2 $0,01$

   1.3 $16\frac{2}{5}$

   1.4 $\sqrt{6\frac{1}{4}}$

   1.5 0

   1.6 2π

   1.7 $-5,3\dot{8}$

   1.8 $\frac{1-\sqrt{2}}{2}$

   1.9 $-\sqrt{-3}$

   1.10 $(\pi)^2$

   1.11 $-\frac{9}{11}$

   1.12 $\sqrt[3]{-8}$

   1.13 $\frac{22}{7}$

   1.14 $2,45897\ldots$

   1.15 $0,\overline{65}$

   1.16 $\sqrt[5]{-32}$

2. Simplify the following:

2.1 $4 \times 4^{2a} \times 4^2 \times 4^a$

2.2 $\frac{3^2}{2^{-3}}$

2.3 $\left(3p^5\right)^2$

2.4 $\frac{k^2k^{3x-4}}{k^x}$

2.5 $\left(5^{(z-1)}\right)^2 + 5^z$

2.6 $\left(\frac{1}{4}\right)^0$

2.7 $\left(x^2\right)^5$

2.8 $\left(\frac{a}{b}\right)^{-2}$

2.9 $(m+n)^{-1}$

2.10 $2\left(p^t\right)^s$

2.11 $\frac{1}{\left(\frac{1}{a}\right)^{-1}}$

2.12 $\frac{k^0}{k^{-1}}$

2.13 $\frac{-2}{-2^{-a}}$

2.14 $\frac{-h}{(-h)^{-3}}$

2.15 $\left(\frac{a^2b^3}{c^3d}\right)^2$

2.16 $10^7\left(7^0\right) \times 10^{-6}(-6)^0 - 6$

2.17 $m^3n^2 \div nm^2 \times \frac{mn}{2}$

2.18 $\left(2^{-2} - 5^{-1}\right)^{-2}$

2.19 $\left(y^2\right)^{-3} \div \left(\frac{x^2}{y^3}\right)^{-1}$

2.20 $\frac{2^{c-5}}{2^{c-8}}$

2.21 $\frac{2^{9a} \times 4^{6a} \times 2^2}{8^{5a}}$

2.22 $\frac{20t^5p^{10}}{10t^4p^9}$

2.23 $\left(\frac{9q^{-2s}}{q^{-3s}y^{-4a-1}}\right)^2$

3. Between which two integers will each of the following roots lie?

3.1 $\sqrt{23}$

3.2 $\sqrt[3]{12}$

3.3 $\sqrt{17}$

3.4 $2\sqrt{26}$

6.2 Exercise 2

1. Simplify the following:

   1.1 $\frac{3^{y-2} \cdot 2^{y+4}}{6^{3y+2}}$

   1.2 $\frac{2^{n} \cdot 8^{n+2}}{16^{n}}$

   1.3 $\frac{25^{x-1}}{125 \cdot 5^{2x-3}}$

   1.4 $\frac{(3 \cdot 3^{x})^{2}}{3^{2x}}$

   1.5 $\frac{75^{x-1}}{5(15)^{x}}$

   1.6 $\frac{4^{p+4} \cdot 12^{p}}{16^{p+1}}$

   1.7 $\frac{10^{2m-3}}{25^{m} \cdot 2^{2m}}$

   1.8 $\frac{3^{\frac{x}{2}} \cdot 9^{\frac{x}{2}}}{3^{1}}$

2. Show that $3^{x} + 3^{x+1} = 4 \cdot 3^{x}$

3. Simplify the following:

   3.1 $\frac{5^{x+1} - 5^{x}}{2 \cdot 5^{x}}$

   3.2 $\frac{2^{x+1} + 2^{x+3}}{10 \cdot 2^{x}}$

   3.3 $\frac{3^{x+1} - 3^{x-1}}{3^{x}}$

   3.4 $\frac{2^{x} + 4^{x+1}}{3 \cdot 2^{x} + 2^{x+1}}$

   3.5 $\frac{49^{x} - 7^{2x-1}}{49^{x}}$

   3.6 $\frac{5^{2} \cdot 5^{x} - 5^{x+1}}{5^{x} + 5^{x+2}}$

4. Simplify the following:

   4.1 $\frac{4^{x} - 1}{2^{x} + 1}$

   4.2 $\frac{16^{x} - 1}{4^{x} - 1}$

   4.3 $\frac{9 - 25^{x}}{3 - 5^{x}}$

   4.4 $\frac{25^{x} - 5^{x}}{5^{x} - 1}$

5. Simplify the following: $\frac{2^{2012} + 2^{2008}}{17}$

6.3 Exercise 3

1. Simplify the following and write answers with positive exponents:

   1.1 $\sqrt{49}$

   1.2 $\sqrt{36^{-1}}$

   1.3 $\sqrt[3]{6^{-2}}$

   1.4 $\sqrt[3]{-\frac{64}{27}}$

   1.5 $\sqrt[4]{(16x^4)^3}$

2. Simplify:

   2.1 $s^{\frac{1}{2}} \div s^{\frac{1}{3}}$

   2.2 $\left(64m^6\right)^{\frac{2}{3}}$

   2.3 $\frac{12m^{\frac{7}{9}}}{8m^{-\frac{11}{9}}}$

   2.4 $(5x)^0 + 5x^0 - (0,25)^{-0,5} + 8^{\frac{2}{3}}$

3. Simplify the following:

   3.1 $\left(\frac{9}{16}\right)^{-\frac{2}{3}}$

   3.2 $\left(4x^2\right)^{\frac{1}{4}}(x)^{\frac{1}{2}}$

   3.3 $\left(r^{\frac{3}{2}}\right) \times \left(4r^{\frac{5}{2}}\right)$

   3.4 $\left(64x^9\right)^{\frac{1}{3}} x^{-1}$

   3.5 $\left(3p^2\right)^{\frac{1}{2}} \cdot \left(25p^{-3}\right)^{\frac{1}{3}}$

6.4 Exercise 4

1. Simplify the following and write answers with positive exponents:

   1.1 $\sqrt[3]{16} \times \sqrt[3]{4}$

   1.2 $\sqrt{a^2b^3} \times \sqrt{b^5c^4}$

   1.3 $\frac{\sqrt{12}}{\sqrt{3}}$

   1.4 $\sqrt{x^2y^{13}} \div \sqrt{y^5}$

2. Simplify the following:

   2.1 $\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$

   2.2 $\frac{b-a}{a^{\frac{1}{2}} - b^{\frac{1}{2}}}$

3. Write in simplest surd form:

   3.1 $\sqrt{72}$

   3.2 $\sqrt{45} + \sqrt{80}$

4. Expand and simplify:

   4.1 $(2 + \sqrt{2})^2$

   4.2 $(2 + \sqrt{2})(1 + \sqrt{8})$

   4.3 $(1 + \sqrt{3})(1 + \sqrt{8} + \sqrt{3})$

5. Prove (without the use of a calculator): $\sqrt{\frac{8}{3}} + 5\sqrt{\frac{5}{3}} - \sqrt{\frac{1}{6}} = \frac{10\sqrt{15}+3\sqrt{6}}{6}$

6. Simplify without the use of a calculator: $\sqrt{5}(\sqrt{45} + 2\sqrt{80})$

6.5 Exercise 5

Rationalise the following:

1. $\frac{10}{\sqrt{5}}$
2. $\frac{3}{\sqrt{6}}$
3. $\frac{2}{\sqrt{3}} \div \frac{\sqrt{2}}{3}$
4. $\frac{3}{\sqrt{5}-1}$
5. $\frac{x}{\sqrt{y}}$
6. $\frac{\sqrt{3}+\sqrt{7}}{\sqrt{2}}$
7. $\frac{3\sqrt{p}-4}{\sqrt{p}}$
8. $\frac{t-4}{\sqrt{t}+2}$
9. $(1 + \sqrt{m})^{-1}$
10. $a(\sqrt{a} \div \sqrt{b})^{-1}$
11. Rationalise the following denominator: $\frac{2}{\sqrt[3]{3}}$
12. Rationalise the following denominator: $\frac{4}{\sqrt[3]{7}}$
13. Rationalise the following denominator: $\frac{2}{\sqrt{8}+1}$
14. Rationalise the following denominator: $\frac{6}{\sqrt{5}+2}$
15. Rationalise the following denominator: $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{6}}$

6.6 Exercise 6

Solve the following equation:

1. $2^x = 2$
2. $4^x = 64$
3. $5^x = 1$
4. $\frac{1}{16} = 4^x$
5. $5 \cdot 4^{x-3} = 320$
6. $(4^x - 16)(3^x - 9) = 0$
7. $\frac{9^x - 1}{3^x - 1} = 2$
8. $8^{8x-1} = (0,125)^{x+3}$
9. $4y^{\frac{3}{2}} - 256 = 0$
10. $\frac{b^{2x} \cdot b^{\frac{1}{x}}}{b^2} = 1$
11. $3^{-3x} \cdot 3^{x+4} = \frac{27^x}{3}$
12. $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0$
13. $4^{2x} + 3 \cdot 4^x - 28 = 0$

6.7 Exercise 7

Solve for the unknown variable (remember to check that the solution is valid):

1. $2^{x+1} - 32 = 0$
2. $125\left(3^p\right) = 27\left(5^p\right)$
3. $2y^{\frac{1}{2}} - 3y^{\frac{1}{4}} + 1 = 0$
4. $t - 1 = \sqrt{7 - t}$
5. $2z - 7\sqrt{z} + 3 = 0$
6. $x^{\frac{1}{3}}\left(x^{\frac{1}{3}} + 1\right) = 6$
7. $2^{4n} - \frac{1}{\sqrt[4]{16}} = 0$
8. $\sqrt{31 - 10d} = 4 - d$

9. $y - 10\sqrt{y} + 9 = 0$
10. $f = 2 + \sqrt{19 - 2f}$
11. $\sqrt{10x + 24} + x + 4 = 0$
12. $\sqrt{x - 3} + 3 = 0$
13. $\sqrt{3 - x} - \sqrt{2x - 4} = 0$
14. $\sqrt{x - 3} \cdot \sqrt{x + 1} = 2\sqrt{3}$

7 ANSWERS FOR EXERCISES

7.1 Exercise 1

1.1 $\mathbb{R}$; $\mathbb{Q}'$

1.2 $\mathbb{R}$; $\mathbb{Q}$

1.3 $\mathbb{R}$; $\mathbb{Q}$

1.4 $\mathbb{R}$; $\mathbb{Q}$

1.5 $\mathbb{R}$; $\mathbb{Q}$; $\mathbb{Z}$; $\mathbb{N}_0$

1.6 $\mathbb{R}$, $\mathbb{Q}'$

1.7 $\mathbb{R}$; $\mathbb{Q}$

1.8 $\mathbb{R}$; $\mathbb{Q}'$

1.9 $\mathbb{R}'$

1.10 $\mathbb{R}$; $\mathbb{Q}'$

1.11 $\mathbb{R}$; $\mathbb{Q}$

1.12 $\mathbb{R}$; $\mathbb{Q}$; $\mathbb{Z}$

1.13 $\mathbb{R}$; $\mathbb{Q}$

1.14 $\mathbb{R}$; $\mathbb{Q}'$

1.15 $\mathbb{R}$; $\mathbb{Q}$

1.16 $\mathbb{R}$; $\mathbb{Q}$; $\mathbb{Z}$

2.1 4^{3a+3}

2.2 72

2.3 $9p^{10}$

2.4 k^{2x-2}

2.5 $5^{2z-2} + 5^z$

2.6 1

2.7 x^{10}

2.8 $\frac{b^2}{a^2}$

2.9 $\frac{1}{m+n}$

2.10 $2p^{ts}$

2.11 $\frac{1}{a}$

2.12 k

2.13 2^{a+1}

2.14 h^4

2.15 $\frac{a^4b^6}{c^6d^2}$

2.16 4

2.17 $\frac{m^2n^2}{2}$

2.18 400

2.19 $\frac{x^2}{y^9}$

2.20 8

2.21 2^{6a+2}

2.22 $2pt$

2.23 $81q^{2s}y^{8a+2}$

3.1 4 and 5

3.2 2 and 3

3.3 4 and 5

3.4 10 and 11

7.2 Exercise 2

1.1 $3^{-2y-4} \cdot 2^{-2y+2}$

1.2 64

1.3 $\frac{1}{25}$

1.4 9

1.5 $\frac{5^{x-3}}{3}$

1.6 $16 \cdot 3^p$

1.7 $\frac{1}{1000}$

1.8 $3^{\frac{3x}{2}-1}$

2 LHS $= 3^x + 3^{x+1}$

$= 3^x + 3^x.3^1$

$= 3^x(1+3)$

$= 4 \cdot 3^x$

$=$ RHS

3.1 2

3.2 1

3.3 $\frac{8}{3}$

3.4 $\frac{1+2^{x+2}}{5}$

3.5 $\frac{6}{7}$

3.6 $\frac{10}{13}$

4.1 $2^x - 1$

4.2 $4^x + 1$

4.3 $3 + 5^x$

4.4 5^x

5 2^{2008}

7.3 Exercise 3

1.1 7

1.2 $\frac{1}{6}$

1.3 $\frac{1}{\sqrt[3]{36}}$

1.4 $-\frac{4}{3}$

1.5 $8x^3$

2.1 $s^{\frac{1}{6}}$

2.2 $16m^4$

2.3 $\frac{3}{2}m^2$

2.4 8

3.1 $\frac{2^{\frac{8}{3}}}{3^{\frac{4}{3}}}$ or $\frac{\sqrt[3]{2^8}}{\sqrt[3]{3^4}}$

3.2 $\sqrt{2}x$ or $2^{\frac{1}{2}}x$

3.3 $4r^4$

3.4 $4x^2$

3.5 $\left(3^{\frac{1}{2}}\right)\cdot\left(5^{\frac{2}{3}}\right)$ or $(\sqrt{3})\cdot\left(\sqrt[3]{5^2}\right)$

7.4 Exercise 4

1.1 4

1.2 ab^4c^2

1.3 2

1.4 xy^4

2.1 $\frac{ab}{b-a}$

2.2 $-\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)$

3.1 $6\sqrt{2}$

3.2 $7\sqrt{5}$

4.1 $6+4\sqrt{2}$

4.2 $6+5\sqrt{2}$

4.3 $4+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$

5 LHS $=\sqrt{\frac{8}{3}}+5\sqrt{\frac{5}{3}}-\sqrt{\frac{1}{6}}$
$=\sqrt{\frac{8}{3}}\times\frac{\sqrt{3}}{\sqrt{3}}+5\sqrt{\frac{5}{3}}\times\frac{\sqrt{3}}{\sqrt{3}}-\sqrt{\frac{1}{6}}\times\frac{\sqrt{6}}{\sqrt{6}}$
$=\frac{\sqrt{8}\sqrt{3}}{3}+5\frac{\sqrt{5}\sqrt{3}}{3}-\frac{\sqrt{6}}{6}$
$=\frac{2\sqrt{24}}{6}+\frac{10\sqrt{15}}{6}-\frac{\sqrt{6}}{6}$
$=\frac{4\sqrt{6}+10\sqrt{15}-\sqrt{6}}{6}$
$=\frac{10\sqrt{15}+3\sqrt{6}}{6}$
$=$ RHS

6 55

7.5 Exercise 5

1. $2\sqrt{5}$
2. $\frac{\sqrt{6}}{2}$
3. $\sqrt{6}$
4. $\frac{3\sqrt{5}+3}{4}$
5. $\frac{x\sqrt{y}}{y}$
6. $\frac{\sqrt{6}+\sqrt{14}}{2}$
7. $\frac{3p-4\sqrt{p}}{p}$
8. $\sqrt{t}-2$
9. $\frac{1-\sqrt{m}}{1-m}$
10. $\sqrt{ab}$
11. $\frac{2\sqrt[3]{9}}{3}$
12. $\frac{4\sqrt[3]{49}}{7}$
13. $\frac{2\sqrt{8}-2}{7}$
14. $6\sqrt{5}-12$
15. $\frac{\sqrt{18}+\sqrt{30}}{6}$

7.6 Exercise 6

1. $x=1$
2. $x=3$
3. $x=0$
4. $x=-2$
5. $x=6$
6. $x=2$
7. $x=0$
8. $x=-\frac{2}{9}$

9. $y = 16$

10. x has no solution

11. $x = 1$

12. $x = 81$ or $x = 16$

13. $x = 1$ or $x \to$ no solution

7.7 Exercise 7

1. $x = 4$

2. $p = 3$

3. $y = 1$ or $y = \frac{1}{16}$

4. $t = 3$

5. $z = \frac{1}{4}$ or $z = 9$

6. $x = -27$ or $x = 8$

7. $n = -\frac{1}{4}$

8. no solution

9. 1 and 81

10. $f = 5$

11. no solution

12. no solution

13. $x = \frac{7}{3}$

14. $x = 5$