CHAPTER 7

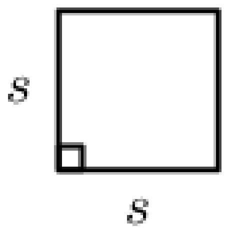
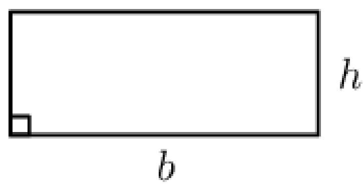
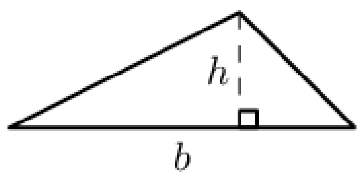
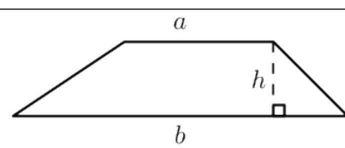
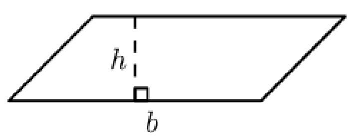
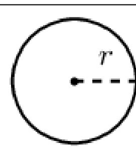
Measurement

CONTENTS

August 26, 2021

This chapter is a revision of perimeters and areas of two dimensional objects and volumes of three dimensional objects.We also examine different combinations of geometric objects and calculate areas and volumes in a variety of real-life contexts.

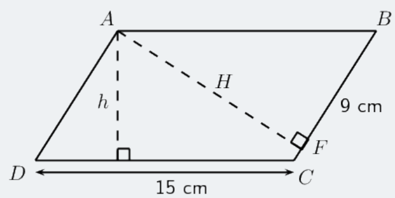
1 AREA OF A POLYGON

Square		Area = s^2
Rectangle		Area = $b \times h$
Triangle		Area = $\frac{1}{2}b \times h$
Trapezium		Area = $\frac{1}{2}(a+b) \times h$
Parallelogram		Area = $b \times h$
Circle		Area = πr^2

Worked example 1: Finding the area of a polygon

Question

$ABCD$ is a parallelogram wit $DC = 15$**cm**, $h = 8cm$ and $BF = 9cm$



Calculate:

1. The area of $ABCD$
2. The perimeter of $ABCD$

Solution

Step 1: Determine the area

The area of a parallelogram $ABCD =$ base $\times$ height:

$$\begin{aligned} \text{Area } &= 15 \times 8 \\ &= 120\text{cm}^2 \end{aligned}$$

Step 2: Determine the perimeter

The perimeter of a parallelogram $ABCD = 2DC + 2BC$

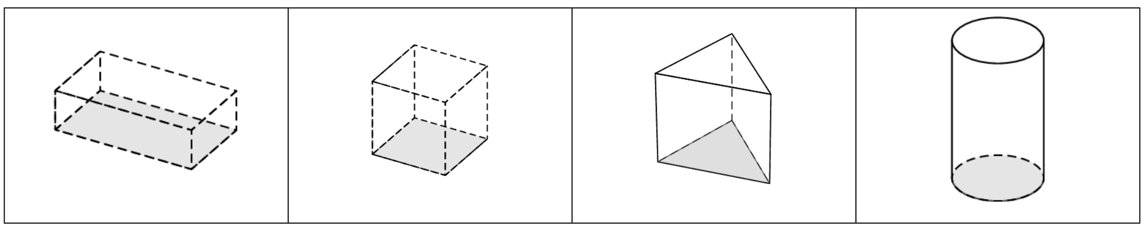
To fine the length of BC, we use $AF \perp BC$ and the theorem of Pythagoras.

$$\begin{aligned} \text{In } \triangle ABF: \quad AF^2 &= AB^2 - BF^2 \\ &= 15^2 - 9^2 \\ &= 144 \\ \therefore AF &= 12\text{cm} \\ \therefore \text{ Area } ABCD &= BC \times AF \\ 120 &= BC \times 12 \\ \therefore BC &= 10\text{cm} \\ \therefore \text{ Perimeter } ABCD &= 2(15) + 2(10) \\ &= 50\text{cm} \end{aligned}$$

2 RIGHT PRISMS AND CYLINDERS

A right prism is a geometric solid that has a polygon as its base and vertical sides perpendicular to the base. The base and top surface are the same shape and size. It is called a "right" prism because the angles between the base and sides are right angles.

A triangular prism has a triangle as its base, a rectangular prism has a rectangle as its base, and a cube is a rectangular prism with all its sides of equal length. A cylinder is another type of right prism which has a circle as its base. Examples of right prisms are given below: a rectangular prism, a cube, a triangular prism and a cylinder.



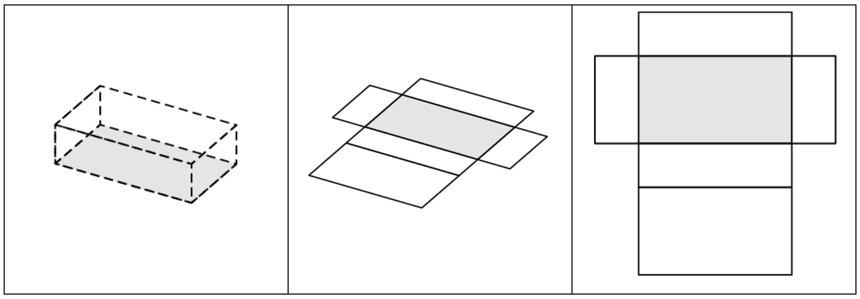
Surface area of prisms and cylinders

Surface area is the total area of the exposed or outer surfaces of a prism. This is easier to understand if we imagine the prism to be a cardboard box that we can unfold. A solid that is unfolded like this is called a net. When a prism is unfolded into a net, we can clearly see each of its faces. In order to calculate the surface area of the prism, we can then simply calculate the area of each face, and add them all together.

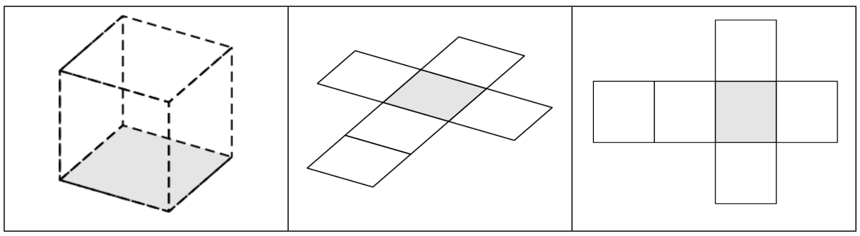
For example, when a triangular prism is unfolded into a net, we can see that it has two faces that are triangles and three faces that are rectangles. To calculate the surface area of the prism, we find the area of each triangle and each rectangle, and add them together.

In the case of a cylinder the top and bottom faces are circles and the curved surface flattens into a rectangle with a length that is equal to the circumference of the circular base. To calculate the surface area we therefore find the area of the two circles and the rectangle and add them together.

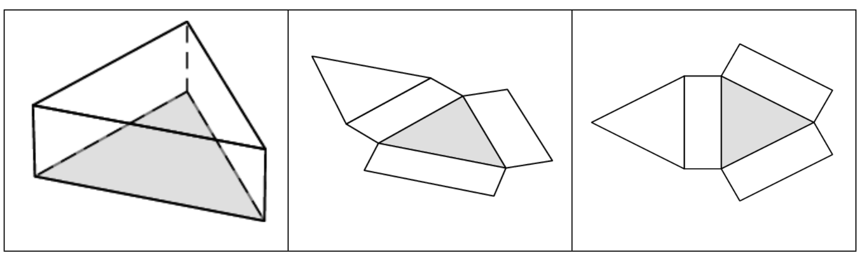
Below are examples of right prisms that have been unfolded into nets. A rectangular prism unfolded into a net is made up of six rectangles.



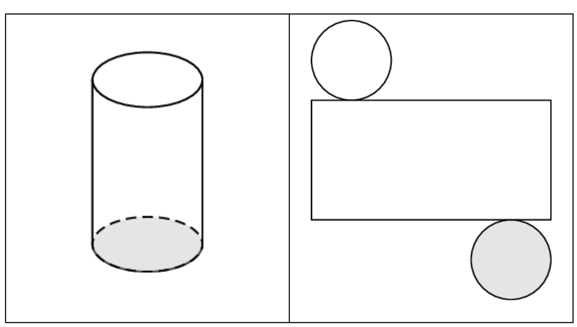
A cube unfolded into a net is made up of six identical squares.



A triangular prism unfolded into a net is made up of two triangles and three rectangles. The sum of the lengths of the rectangles is equal to the perimeter of the triangles.



A cylinder unfolded into a net is made up of two identical circles and a rectangle with length equal to the circumference of the circles.

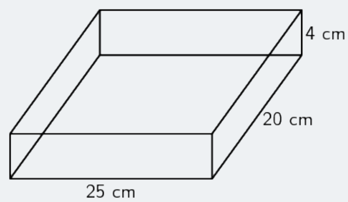


Worked example 2: Calculating surface area

Question

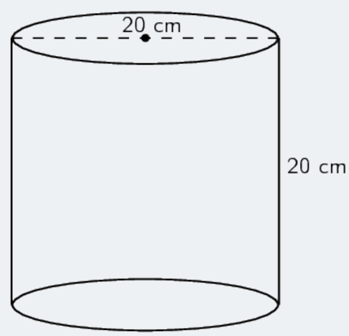
A box of chocolates has the following dimensions:

$$\text{length} = 25\text{cm}$$
$$\text{width} = 20\text{cm}$$
$$\text{height} = 4\text{cm}$$



And a cylindrical tin of biscuits has the following dimensions:

$$\text{diameter} = 20\text{cm}$$
$$\text{width} = 20\text{cm}$$



1. Calculate the area of the wrapping paper needed to cover the entire box (assume no overlapping at the corners).
2. Determine if this same sheet of wrapping paper would be enough to cover the tin of biscuits.

Solution

Step 1: Determine the area of the rectangular box

$$\begin{aligned}\text{Surface area} &= 2 \times (25 \times 20) + 2 \times (20 \times 4) + 2 \times (25 \times 4) \\ &= 1360\text{cm}^2\end{aligned}$$

Worked example 2 continued

Step 2: Determine the area of the cylindrical tin

The radius of the cylinder = $\frac{20}{2}$ = 10cm

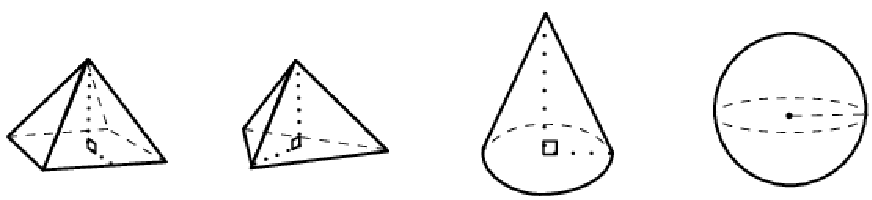
$$\begin{aligned}\text{Surface area} &= 2 \times \pi(10)^2 + 2\pi(10)(20) \\ &= 1885\text{cm}^2\end{aligned}$$

Step 3: Write the final answer

No, the area of the sheet of wrapping paper used to cover the box is not big enough to cover the tin.

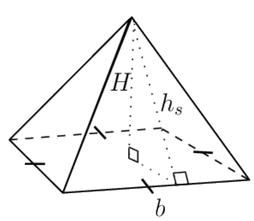
3 RIGHT PYRAMIDS, RIGHT CONES AND SPHERES

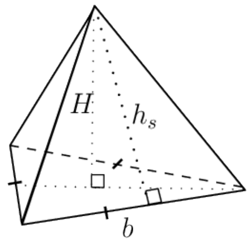
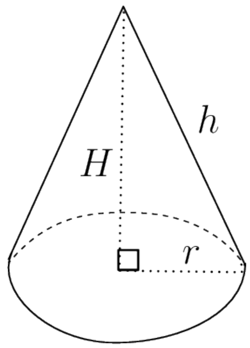
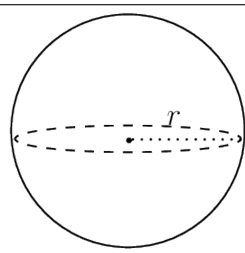
A pyramid is a geometric solid that has a polygon as its base and sides that converge at a point called the apex. In other words the sides are not perpendicular to the base.



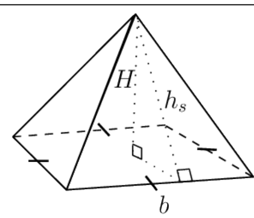
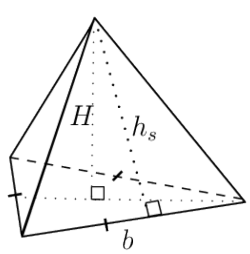
The triangular pyramid and square pyramid take their names from the shape of their base. We call a pyramid a "right pyramid" if the line between the apex and the centre of the base is perpendicular to the base. Cones are similar to pyramids except that their bases are circles instead of polygons. Spheres are solids that are perfectly round and look the same from any direction.

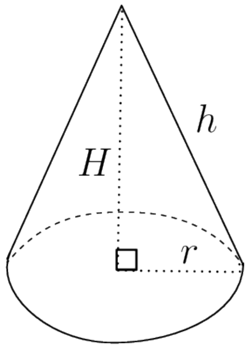
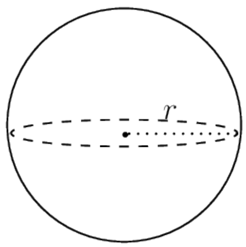
Surface area of pyramids, cones and spheres

Square pyramid		$\begin{aligned}\text{Surface area} &= \text{area of base} + \text{area of triangular sides} \\ &= b^2 + 4(\frac{1}{2}bh_s) \\ &= b(b + 2h_s)\end{aligned}$

Triangular pyramid		Surface area = area of base + area of triangular sides $= (\frac{1}{2}b \times h_b) + 3(\frac{1}{2}b \times h_s)$ $= \frac{1}{2}b(h_b + 3h_s)$
Right cone		Surface area = area of base + area of walls $= \pi r^2 + fra$ $= \frac{1}{2}b(h_b + 3h_s)$
Sphere		Surface area $= 4\pi r^2$

Volume of pyramids, cones and spheres

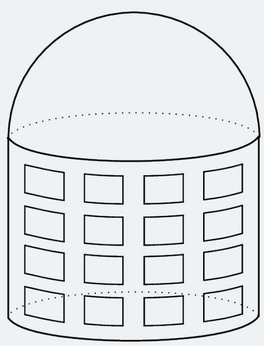
Square pyramid		Volume $= \frac{1}{3} \times$ area of base $\times$ height of pyramid $= \frac{1}{3} \times b^2 \times H$
Triangular pyramid		Volume $= \frac{1}{3} \times$ area of base $\times$ height of pyramid $= \frac{1}{3} \times \frac{1}{2}bh \times H$ $= \frac{1}{2}b(h_b + 3h_s)$

Right cone		Volume $= \frac{1}{3} \times$ area of base $\times$ height of cone $= \pi r^2 + fra$ $= \frac{1}{3} \times \pi r^2 \times H$
Sphere		Volume $= \frac{4}{3}\pi r^3$

Worked example 3: Finding surface area and volume

Question

The Southern African Large Telescope (SALT) is housed in a cylindrical building with a domed roof in the shape of a hemisphere. The height of the building wall is 17m and the diameter is 26m.



1. Calculate the total surface area of the building.
2. Calculate the total volume of the building.

Worked example 3 continued

Solution

Step 1: Calculate the total surface area

$$\begin{aligned} \text{Total surface area} &= \text{area of the dome} + \text{area of the cylinder} \\ \text{Surface area} &= [\frac{1}{2}(4\pi r^2)] + [2\pi r \times h] \\ &= \frac{1}{2}(4\pi)(13)^2 + 2\pi(13)(17) \\ &= 2450\text{m}^2 \end{aligned}$$

Step 2: Calculate the total volume

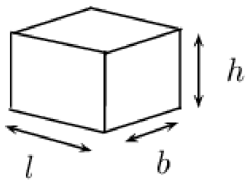
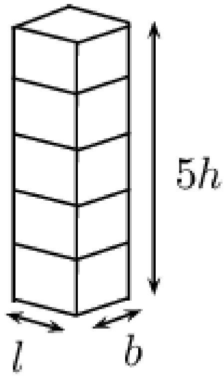
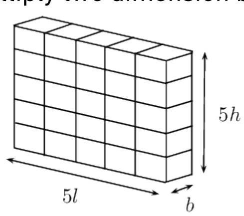
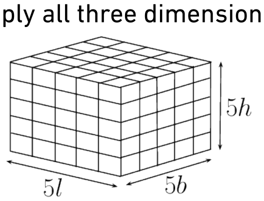
$$\begin{aligned} \text{Total volume} &= \text{volume of the dome} + \text{volume of the cylinder} \\ \text{Volume} &= [\frac{1}{2} \times (\frac{4}{3}\pi r^3)] + [\pi r^2 \times h] \\ &= \frac{2}{3}\pi(13)^3 + \pi(11)^2(17) \\ &= 9543\text{m}^2 \end{aligned}$$

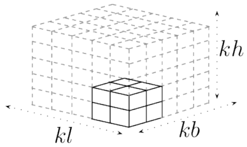
4 MULTIPLYING A DIMENSION BY A CONSTANT FACTOR

When one or more of the dimensions of a prism or cylinder is multiplied by a constant, the surface area and volume will change. The new surface area and volume can be calculated by using the formulae from the preceding section.

It is important to see a relationship between the change in dimensions and the resulting change in surface area and volume. These relationships make it simpler to calculate the new volume or surface area of an object when its dimensions are scaled up or down.

Consider a rectangular prism of dimensions ***l***, ***b*** and ***h***. Below we multiply one, two and three of its dimensions by a constant factor of 5 and calculate the new volume and surface area.

Dimensions	Volume	Surface
Original dimensions 	$V = l \times b \times h$ $= lbh$	$A = 2[(l \times h) + (l \times b) + (b \times h)]$ $= 2(lh + lb + bh)$
Multiply one dimension by 5 	$V = l \times b \times 5h$ $= 5(lbh)$ $= 5V$	$A_1 = 2[(l \times 5h) + (l \times b) + (b \times 5h)]$ $= 2(5lh + lb + 5bh)$
Multiply two dimension by 5 	$V = 5l \times b \times 5h$ $= 5 \cdot 5(lbh)$ $= 5^2 V$	$A_2 = 2[(5l \times 5h) + (5l \times b) + (b \times 5h)]$ $= 2 \times 5(5lh + lb + 5bh)$
Multiply all three dimension by 5 	$V = 5l \times 5b \times 5h$ $= 5^3(lbh)$ $= 5^3 V$	$A_3 = 2[(5l \times 5h) + (5l \times 5b) + (5b \times 5h)]$ $= 2 \times (5^2 lh + 5^2 lb + 5^2 bh)$ $= 5^2 \times 2(lh + lb + bh)$ $= 5^2 A$

Multiply all three dimension by **k** 	$V = kl \times kb \times kh$ $= k^3(lbh)$ $= k^3V$	$A_k = 2[(kl \times kh) + (kl \times kb) + (kb \times kh)]$ $= 2 \times (k^2lh + k^2lb + k^2bh)$ $= k^2 \times 2(lh + lb + bh)$ $= k^2A$

Worked example 4: The effects of k

Question

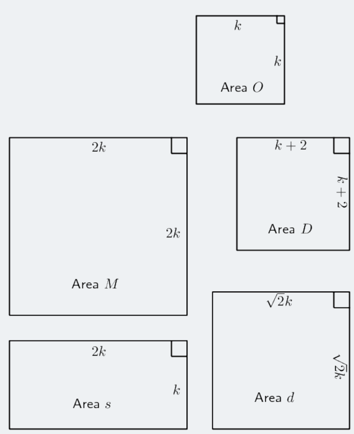
The Nash family wants to build a television room onto their house. The dad draws up the plans for the new square room of length k metres. The mum looks at the plans and decides that the area of the room needs to be doubled. To achieve this:

- the mum suggests doubling the length of the sides of the room
- the dad recommends adding 2m to the length of the sides
- the daughter suggests multiplying the length of the sides by a factor of $\sqrt{2}$
- the son suggests doubling only the width of the room

Who's suggestion will double the area of the square room? Show all calculations.

Solution

Step 1: Draw a sketch



Worked example 4 continued

Step 2: Calculate and compare

First calculate the area of the square room in the original plan:

$$\begin{aligned} \text{Area}0 &= \text{length} \times \text{length} \\ &= k^2 \end{aligned}$$

Therefore, double the area of the room would be 2k^2.

Consider the mum's suggestion of doubling the length of the sides of the room:

$$\begin{aligned} \text{Area M} &= \text{length} \times \text{length} \\ &= 2k \times 2k \\ &= 4k^2 \end{aligned}$$

This area would be 4 times the original area.

The dad suggests adding 2m to the length of the sides of the room:

$$\begin{aligned} \text{Area D} &= \text{length} \times \text{length} \\ &= (k+2) \times (k+2) \\ &= k^2 + 4k + 2 \\ &\neq 2k^2 \end{aligned}$$

This is not double the original area.

The daughter suggests multiplying the length of the sides by a factor of $\sqrt{2}$:

$$\begin{aligned} \text{Area d} &= \text{length} \times \text{length} \\ &= \sqrt{2}k \times \sqrt{2}k \\ &= 2k^2 \end{aligned}$$

The daughter's suggestion would double the area of the room. Practically, the length of the room could be multiplied by $\sqrt{2} \approx 1,41$ which would given an area of $1,96\text{m}^2$.

The son suggests doubling only the width of the room:

$$\begin{aligned} \text{Area s} &= \text{length} \times \text{length} \\ &= 2k \times k \\ &= 2k^2 \end{aligned}$$

The son's suggestion would double the area of the room, however the room would no longer be a square.

Step 3: Write the final answer

The daughter's suggestion of multiplying the length of the sides of the room by a factor of $\sqrt{2}$ would keep the shape of the room a square and would double the area of the room.

5 SUMMARY

- Area is the two dimensional space inside the boundary of a flat object.
- Area formulae:
  - square: s^2
  - rectangle: $b \times h$
  - triangle: $\frac{1}{2}b \times h$
  - trapezium: $\frac{1}{2}(a+b) \times h$
  - parallelogram: $b \times h$
  - circle: πr^2
- Surface area is the total area of the exposed or outer surfaces of a prism.
- A net is the unfolded "plan" of a solid.
- Volume is the three dimensional space occupied by an object, or the contents of an object.
  - Volume of a rectangular prism: $l \times b \times h$
  - Volume of a triangular prism: $\frac{1}{2}b \times h) \times H$
  - Volume of a square prism or cube: s^3
  - Volume of a cylinder: $\pi r^2 \times h$
- A pyramid is a geometric solid that has a polygon as its base and sides that converge at a point called the apex. The sides are not perpendicular to the base.
- Surface area formulae:
  - square pyramid: $b(b+2h)$
  - triangular pyramid: $\frac{1}{2}b(h_b + 3h_s)$
  - right cone: $\pi r(r + h_s)$
  - sphere: $4\pi r^2$
- Volume formulae:
  - square pyramid: $\frac{1}{3} \times b^2 \times H$
  - triangular pyramid: $\frac{1}{3} \times \frac{1}{2}bh \times H$
  - right cone: $\frac{1}{3} \times \pi r^2 \times H$
  - sphere: $\frac{4}{3}\pi r^3$

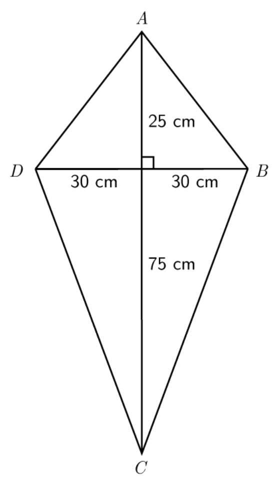
6 EXERCISES

6.1 Exercise 1

1. Vuyo and Banele are having a competition to see who can build the best kite using balsa wood (a lightweight wood) and paper. Vuyo decides to make his kite with one diagonal 1 m long and the other diagonal 60 cm long. The intersection of the two diagonals cuts the longer diagonal in the ratio $1:3$.

   Banele also uses diagonals of length 60 cm and 1 m, but he designs his kite to be rhombus-shaped.

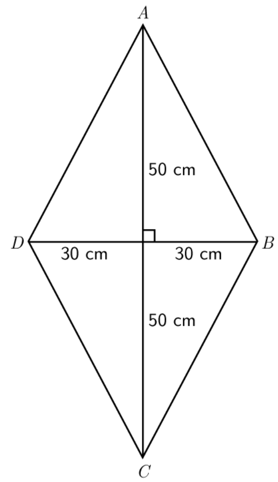
   1.1 In the following sketch of Vuyo's kite, write down the length of the vertical support beam AC in cm to the nearest integer.



   1.2 Determine how much balsa wood Vuyo will need to build the outside frame of the kite to the nearest cm.

   1.3 Calculate how much paper is needed to cover the frame of the kite in m^2 to one decimal place.

1.4 In the following sketch of Banele's kite, write down the length of the vertical support beam AC in cm to the nearest integer.



1.5 Determine the amount of wood needed for Banele's kite in cm to 1 decimal place.

1.6 Determine the amount of paper needed for Banele's kite in cm^2 to 1 decimal place.

1.7 Which design uses the least amount of balsa wood?

2. O is the centre of the bigger semi-circle with a radius of 10 units. Two smaller semi-circles are inscribed into the bigger one, as shown on the diagram. Calculate the following (in terms of):

   2.1 The area of the shaded figure.

   2.2 The perimeter encloses the shaded area.

3. Karen's engineering textbook is $30\ cm$ long and $20\ cm$ wide. She notices that the dimensions of her desk are in the same proportion as the dimensions of her textbook.

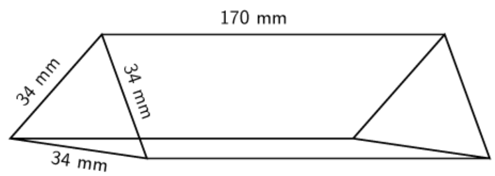
   3.1 If the desk is $90\ cm$ wide, calculate the area of the top of the desk in m^2 to one decimal place.

   3.2 Karen uses some cardboard to cover each corner of her desk with an isosceles triangle, as shown in the following diagram. Calculate the new perimeter and area of the visible part of the top of her desk.

   3.3 Use this new area to calculate the dimensions of a square desk with the same desk top area.

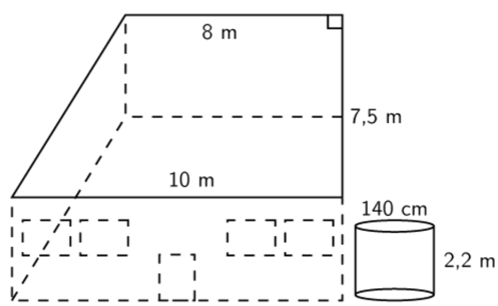
6.2 Exercise 2

1. A popular chocolate container is an equilateral right triangular prism with sides of $34\ mm$. The box is $170\ mm$ long. Calculate the surface area of the box (to the nearest square centimetre).



2. Gordon buys a cylindrical water tank to catch rain water off his roof. He discovers a full $2\ l$ tin of green paint in his garage and decides to paint the tank (not the base). If he uses $250\ ml$ to cover $1\ m^2$, will he have enough green paint to cover the tank with one layer of paint? The diameter and height of the tank are $1,1\ m$ and $1,4\ m$ respectively.

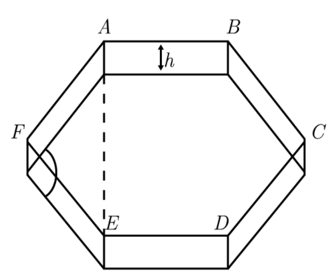
3. The roof of Phumza's house is the shape of a right-angled trapezium. A cylindrical water tank is positioned next to the house so that the rain on the roof runs into the tank. The diameter of the tank is $140\ cm$ and the height is $2,2\ m$.



   3.1 Determine the area of the roof in m^2 to 1 decimal place.

   3.2 Determine how many litres of water the tank can hold to 2 decimal places.

4. The length of a side of a hexagonal sweet tin is $8\ cm$ and its height is equal to half of the side length.



4.1 Show that the interior angles are equal to $120°$.

4.2 Determine the length of the line AE in cm to 2 decimal places.

4.3 Calculate the volume of the tin in cm^3 to 2 decimal places.

6.3 Exercise 3

1. An ice-cream cone has a diameter of $52,4\ mm$ and a total height of $146\ mm$.

   1.1 Calculate the surface area of the ice-cream and the cone to the nearest square centimeter.

   1.2 Calculate the total volume of the ice-cream and the cone to the nearest cubic centimeter.

   1.3 How many ice-cream cones can be made from a $5l$ tub of ice-cream (assume the cone is completely filled with ice-cream)?

   1.4 Consider the net of the cone given below. R is the length from the tip of the cone to its perimeter, P.

      i. Determine the value of R.

      ii. Calculate the length of arc P.

      iii. Determine the length of arc M.

2. Complete the following sentences:

   2.1 If one dimension of a cube is multiplied by a factor $\frac{1}{2}$, the volume of the cube...

   2.2 If two dimensions of a cube are multiplied by a factor 7, the volume of the cube...

   2.3 If three dimensions of a cube are multiplied by a factor 3, then:

      i. each side of the cube will...

      ii. the outer surface area of the cube will...

      iii. the volume of the cube will...

   2.4 If each side of a cube is halved, then:

      i. the outer surface area of the cube will...

      ii. the volume of the cube will...

3. The municipality intends building a swimming pool of volume W^3 cubic metres. However, they realise that it will be very expensive to fill the pool with water, so they decide to make the pool smaller.

   3.1 The length and breadth of the pool are reduced by a factor of $\frac{7}{10}$. Express the new volume in terms of W.

   3.2 The dimensions of the pool are reduced so that the volume of the pool decreases by a factor of $0,8$. Determine the new dimensions of the pool in terms of W (remember that the pool must be a cube).

7 ANSWERS FOR EXERCISES

7.1 Exercise 1

1.1 $100\ cm$

1.2 $240\ cm$

1.3 $0,6\ cm$

1.4 $100\ cm$

1.5 $233,2\ cm$

1.6 $0,6\ cm$

1.7 Banele's design uses the least amount of wood.

2.1 25π units^2

2.2 20π units^2

3.1 $1,2\ m^2$

3.2 New perimeter = $414,8\ cm$
New area = $11700\ cm^2$

3.3 $\approx 108 \times 108\ cm^2$

7.2 Exercise 2

1. $273\ cm^2$

2. Yes, he will have enough paint for 1 layer.

3.1 $67,5\ m^2$

3.2 $3,39\ l$

4.1 $120°$

4.2 $13,86\ cm$

4.3 $554,24\ m^2$

7.3 Exercise 3

1.1 $120\,cm^2$

1.2 $124\,cm^2$

1.3 $40\,cones$

1.4 $120\,mm$, $165\,mm$, $589\,mm$

2.1 Is halved

2.2 Becomes 49 times larger

2.3 Become 3 times larger; Become $3^2 = 9$ times bigger; Become $3^3 = 27$

2.4 i. be multiplied by a factor of $\frac{1}{2}^2$ = $\frac{1}{4}$, therefore surface area become 4 times smaller.
 ii. be multiplied by a factor of $\frac{1}{2}^3$ = $\frac{1}{8}$, therefore surface area become 8 times smaller.

3.1 $0,49W$

3.2 $0,93 \times W$