CHAPTER 3

Finance

CONTENTS

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In earlier grades we studied simple interest and compound interest, together with the concept of depreciation. Nominal and effective interest rates were also described.

- Simple interest: $A = P(1 + in)$
- Compound interest: $A = P(1 + i)^n$
- Simple depreciation: $A = P(1 - in)$
- Compound depreciation: $A = P(1 - i)^n$
- Nominal and effective annual interest rates: $1 + i = (1 + \frac{i^{(m)}}{m})^m$

In this chapter we look at different types of annuities, sinking funds and pyramid schemes. We also look at how to critically analyse investment and loan options and how to make informed financial decisions.

Financial planning is very important and it allows people to achieve certain goals, such as supporting a family, attending university, buying a house, and saving enough money for retirement. Prudent financial planning includes making a budget, opening a savings account, wisely investing savings and planning for retirement.

1 CALCULATING THE PERIOD OF AN INVESTMENT

For calculations using the simple interest formula, we solve for n, the time period of an investment or loan, by simply rearranging the formula to make n the subject. For compound interest calculations, where n is an exponent in the formula, we need to use our knowledge of logarithms to determine the value of n.]

$$\begin{aligned} A &= P(1+i)^n \\ A &= \text{accumulated amount} \\ P &= \text{principal amount} \\ i &= \text{interest rate written as a decimal} \\ n &= \text{time period} \end{aligned}$$

Solving for n:

$$\begin{aligned} A &= P(1+i)^n \\ \frac{A}{P} &= (1+i)^n \\ \text{Use definition: } n &= \log_{(1+i)} \frac{A}{P} \\ \text{Change of base: } n &= \frac{\log(\frac{A}{P})}{\log(1+i)} \end{aligned}$$

WORKED EXAMPLE 1: DETERMINING THE VALUE OF n

QUESTION

Thembile invests $R3\ 500$ into a savings account which pays $7,5$% per annum compounded yearly. After an unknown period of time his account is worth $R4\ 044,69$. For how long did Thembile invest his money?

SOLUTION

Step 1: Write down the compound interest formula and the unknown values

$$A = P(1+i)^n$$
$$A = 4\ 044.69$$
$$P = 3\ 500$$
$$i = 0.075$$

Step 2: Substitute the values and solve for n

$$\begin{aligned} A &= P(1+i)^n \\ 4\ 044.69 &= 3\ 500(1+0.075)^n \\ \frac{4\ 044.69}{3\ 500} &= 1.075^n \\ \therefore n &= \log_{1.075} \frac{4\ 044.69}{3\ 500} \\ &= \frac{\log \frac{4\ 044.69}{3\ 500}}{\log 1.075} \\ &= 2.00\ldots \end{aligned}$$

Step 3: Write final answer

The $R3500$ was invested for 2 years.

WORKED EXAMPLE 2: DURATION OF INVESTMENTS

QUESTION

Margo has $R12000$ to invest and needs the money to grow to at least $R30000$ to pay for her daughter's studies. If it is invested at a compound interest rate of 9% per annum, determine how long (in full years) her money must be invested?

SOLUTION

Step 1: Write down the compound interest formula and the unknown values

$$A = P(1+i)^n$$
$$A = 30\ 000$$
$$P = 12\ 000$$
$$i = 0.09$$

Step 2: Substitute the values and solve for n

$$\begin{aligned} A &= P(1+i)^n \\ 30\ 000 &= 12\ 000(1+0,09)^n \\ \frac{5}{2} &= (1.09)^n \\ \therefore n &= \log_{1.09}\frac{5}{2} \text{ (use definition)} \\ &= \frac{\log\frac{5}{2}}{\log 1.09} \text{ (change of base)} \\ &= 10.632\ldots \end{aligned}$$

Step 3: Write the final answer

In this case we round up, because 10 years will not yet deliver the required $R30000$. Therefore the money must be invested for at least 11 years.

2 ANNUITIES

Definition

Annuity

A number of equal payments made at regular intervals for a certain amount of time. An annuity is subject to a rate of interest.

- **Future value annuity** - regular equal deposits/payments are made into a savings account or investment fund to provide an accumulated amount at the end of the time period. The amount accumulating in the fund earns compound interest at a certain rate.
- **Present value annuity** - regular equal payments/installments are made to pay back a loan or bond over a given time period. The reducing balance of the loan is usually charged compound interest at a certain rate.

For investment funds, pension funds, loan repayments, mortgage bonds (home loan) and other types of annuities, payments are typically made each month. To "default" on a payment means that a payment for a certain month was not paid. The period of an investment is also referred to as the term of an investment.

3 FUTURE VALUE ANNUITIES

For future value annuities, we regularly save the same amount of money into an account, which earns a certain rate of compound interest, so that we have money for the future.

WORKED EXAMPLE 3: FUTURE VALUE ANNUITIES

QUESTION

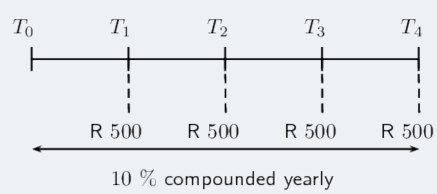
At the end of each year for 4 years, Kobus deposits $R500$ into an investment account. If the interest rate on the account is 10% per annum compounded yearly, determine the value of his investment at the end of the 4 years.

SOLUTION

Step 1: Write down the given information and the compound interest formula

$$A = P(1+i)^n$$
$$P = 500$$
$$i = 0.1$$
$$n = 4$$

Step 2: Draw a timeline



The first deposit in the account earns the highest amount of interest (three interest payments) and the last deposit earns the least interest (no interest payments).

We can summarize this information in the table below:

	Deposit	No. of interest payments	Calculation	Accumulated amount
Year 1	$R500$	3	$500(1+0.1)^3$	$R665.50$
Year 2	$R500$	2	$500(1+0.1)^2$	$R605.00$
Year 3	$R500$	1	$500(1+0.1)^1$	$R550.00$
Year 4	$R500$	0	$500(1+0.1)^0$	$R500.00$
Total				$R2320.50$

3.1 Deriving the formula

Note: for this section is it important to be familiar with the formulae for the sum of a geometric series (Chapter 1):

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ (for } r > 1\text{)}$$
$$S_n = \frac{a(1 - r^n)}{1 - r} \text{ (for r <1)}$$

In the worked example above, the total value of Kobus' investment at the end of the four year period is calculated by summing the accumulated amount for each deposit:

$$\begin{aligned} R2\,320.50 &= R500.00 + R550.00 + R605.00 + R665,50 \\ &= 500(1 + 0.1)^0 + 500(1 + 0.1)^1 + 500(1 + 0.1)^2 + 500(1 + 0.1)^3 \end{aligned}$$

We notice that this is a geometric series with a constant ratio $r = 1 + 0.1$.
Using the formula for the sum of a geometric series:

$$\begin{aligned} a &= 500 \\ r &= 1.1 \\ n &= 4 \\ S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{500(1.1^4 - 1)}{1.1 - 1} \\ &= 2\,320.50 \end{aligned}$$

We can therefore use the formula for the sum of a geometric series to derive a formula for the future value (F) of a series of (n) regular payments of an amount (x) which are subject to an interest rate (i):

$$\begin{aligned} a &= x \\ r &= 1 + i \\ S_n &= \frac{a(r^n - 1)}{r - 1} \\ \therefore F &= \frac{x[(1 + i)^n - 1]}{(1 + i) - 1} \\ &= \frac{x[(1 + i)^n - 1]}{i} \end{aligned}$$

Future value of payments:

$$F = \frac{x[(1 + i)^n - 1}{i}$$

If we are given the future value of a series of payments, then we can calculate the value of the payments by making x the subject of the above formula.

Payment amount:

$$x = \frac{F \times i}{[(1+i)^n - 1]}$$

WORKED EXAMPLE 4: FUTURE VALUE ANNUITIES

QUESTION

Ciza decides to start saving money for the future. At the end of each month she deposits $R900$ into an account at Harringstone Mutual Bank, which earns $8,25\%$ interest p.a. compounded monthly.

1. Determine the balance of Ciza's account after 29 years.
2. How much money did Ciza deposit into her account over the 29 year period?
3. Calculate how much interest she earned over the 29 year period.

SOLUTION

Step 1: Write down the given information and the future value formula

$$\begin{aligned} F &= \frac{x[(1+i)^n - 1]}{i} \\ x &= 900 \\ i &= \frac{0.0825}{12} \\ n &= 29 \times 12 = 348 \end{aligned}$$

Step 2: Substitute the known values and use a calculator to determine F

$$\begin{aligned} F &= \frac{900[(1+\frac{0.0825}{12})^{348} - 1]}{\frac{0.0825}{12}} \\ &= R1\ 289\ 665.06 \end{aligned}$$

Remember: do not round off at any of the interim steps of a calculation as this will affect the accuracy of the final answer.

Step 3: Calculate the total value of deposits into the account

Ciza deposited $R900$ each month for 29 years:

$$\begin{aligned} \text{Total deposits} &= R900 \times 12 \times 29 \\ &= R313\ 200 \end{aligned}$$

Step 4: Calculate the total interest earned

$$\begin{aligned} \text{Total interest} &= \text{final account balance} - \text{total value of all deposits} \\ &= R1\ 289\ 665.06 - R313\ 200 \\ &= R976\ 465.06 \end{aligned}$$

3.2 Useful tips for solving problems:

1. Timelines are very useful for summarising the given information in a visual way.
2. When payments are made more than once per annum, we determine the total number of payments (n) by multiplying the number of years by p:

Term	p
yearly/annually	1
half-yearly / bi-annually	2
quarterly	4
monthly	12
weekly	52
daily	365

3. If a nominal interest rate $(i^{(m)})$ is given, then use the following formula to convert it to an effective interest rate:

$$1 + i = (1 + \frac{i^{(m)}}{m})^m$$

WORKED EXAMPLE 5: CALCULATING THE MONTHLY PAYMENTS

QUESTION

Kosma is planning a trip to Canada to visit her friend in two years' time. She makes an itinerary for her holiday and she expects that the trip will cost $R25000$. How much must she save at the end of every month if her savings account earns an interest rate of $10,7\%$ per annum compounded monthly?

SOLUTION

Step 1: Write down the given information and the future value formula

$$F = \frac{x[(1+i)^n - 1}{i}$$

To determine the monthly payment amount, we make x the subject of the formula:

$$x = \frac{F \times i}{[(1+i)^n - 1]}$$

$$F = 2\ 5000$$

$$i = \frac{0.107}{12}$$

$$n = 2 \times 12 = 24$$

Step 2: Substitute the known values and calculate x

$$x = \frac{2\ 5000 \times \frac{0.107}{12}}{[(1+\frac{0.107}{12})^{24} - 1]}$$

$$= R938.80$$

Step 3: Write the final answer

Kosma must save $R938,80$ each month so that she can afford her holiday.

WORKED EXAMPLE 6: DETERMINING THE VALUE OF AN INVESTMENT

QUESTION

Simon starts to save for his retirement. He opens an investment account and immediately deposits $R800$ into the account, which earns $12,5\%$ per annum compounded monthly. Thereafter, he deposits $R800$ at the end of each month for 20 years. What is the value of his retirement savings at the end of the 20 year period?

SOLUTION

Step 1: Write down the given information and the future value formula

$$F = \frac{x[(1+i)^n - 1]}{i}$$
$$x = 800$$
$$i = \frac{0.125}{12}$$
$$n = 1 + (20 \times 12) = 241$$

Note that we added one extra month to the 20 years because Simon deposited $R800$ immediately.

Step 2: Substitute the known values and calculate F

$$F = \frac{800[(1+\frac{0.125}{12})^{241} - 1)}{\frac{0.125}{12}}$$
$$= R856\,415.66$$

Step 3: Write the final answer

Simon will have saved $R856415,66$ for his retirement.

3.3 Sinking funds

Vehicles, equipment, machinery and other similar assets, all depreciate in value as a result of usage and age. Businesses often set aside money for replacing outdated equipment or old vehicles in accounts called sinking funds. Regular deposits, and sometimes lump sum deposits, are made into these accounts so that enough money will have accumulated by the time a new machine or vehicle needs to be purchased.

WORKED EXAMPLE 7: SINKING FUNDS

QUESTION

Wellington Courier Company buys a delivery truck for $R296000$. The value of the truck depreciates on a reducing-balance basis at 18% per annum. The company plans to replace this truck in seven years' time and they expect the price of a new truck to increase annually by 9%.

1. Calculate the book value of the delivery truck in seven years' time.
2. Determine the minimum balance of the sinking fund in order for the company to afford a new truck in seven years' time.
3. Calculate the required monthly deposits if the sinking fund earns an interest rate of 13% per annum compounded monthly.

SOLUTION

Step 1: Determine the book value of the truck in seven years' time

$$\begin{aligned} P &= 296\ 000 \\ i &= 0.18 \\ n &= 7 \\ A &= P(1-i)^n \\ &= 296\ 000(1-0.18)^7 \\ &= R73\ 788.50 \end{aligned}$$

Step 2: Determine the minimum balance of the sinking fund

Calculate the price of a new truck in seven years' time:

$$\begin{aligned} P &= 296\ 000 \\ i &= 0.09 \\ n &= 7 \\ A &= P(1+i)^n \\ &= 296\ 000(1+0.09)^7 \\ &= R541\ 099.58 \end{aligned}$$

Therefore, the balance of the sinking fund (F) must be greater than the cost of a new truck in seven years' time minus the money from the sale of the old truck:

$$\begin{aligned} F &= R541\ 099.58 - R73\ 788.5 \\ &= R467\ 311.08 \end{aligned}$$

Step 3: Calculate the required monthly payment into the sinking fund

$$x = \frac{F \times i}{[(1+i)^n - 1]}$$
$$F = 467\ 311.08$$
$$i = \frac{0.13}{12}$$
$$n = 7 \times 12 = 83$$

Substitute the values and calculate x:

$$x = \frac{467\ 311.08 \times \frac{0.13}{12}}{[(1+\frac{0.13}{12})^{84} - 1]}$$
$$= R3\ 438.77$$

Therefore, the company must deposit $R3\ 438,77$ each month.

4 PRESENT VALUE ANNUITIES

For present value annuities, regular equal payments/installments are made to pay back a loan or bond over a given time period. The reducing balance of the loan is usually charged compound interest at a certain rate. In this section we learn how to determine the present value of a series of payments.

Consider the following example:

Kate needs to withdraw $R1000$ from her bank account every year for the next three years. How much must she deposit into her account, which earns 10% per annum, to be able to make these withdrawals in the future? We will assume that these are the only withdrawals and that there are no bank charges on her account.
To calculate Kate's deposit, we make P the subject of the compound interest formula:

$$A = P(1+i)^n$$
$$\frac{A}{(1+i)^n} = P$$
$$\therefore P = A(1+i)^{-n}$$

We determine how much Kate must deposit for the first withdrawal:

$$P = 1\ 000(1+0.1)^{-1}$$
$$= 909.09$$

We repeat this calculation to determine how much must be deposited for the second and third withdrawals:

$$\begin{aligned}\text{Second withdrawal: } P &= 1\,000(1+0.1)^{-2} \\ &= 826.45 \\ \text{Third withdrawal: } P &= 1\,000(1+0.1)^{-3} \\ &= 751.31\end{aligned}$$

Notice that for each year's withdrawal, the deposit required gets smaller and smaller because it will be in the bank account for longer and therefore earn more interest. Therefore, the total amount is:

$$R909.09 + R826.45 + R751.31 = R2\,486.85$$

We can check these calculations by determining the accumulated amount in Kate's bank account after each withdrawal:

	Calculation	Accumulated amount
Initial deposit		$R2\,486.85$
Amount after one year	$= 2\,486.85(1+0.1)$	$= R2\,735.54$
Amount after first withdrawal	$= R2\,735.54 - R1\,000$	$= R1\,735.54$
Amount after two years	$= 1\,735.54(1+0.1)$	$= 1\,909.09$
Amount after second withdrawal	$= R1\,909.09 - R1\,000$	$= 909.09$
Amount after three years	$= 909.09(1+0.1)$	$R1\,000$
Amount after third withdrawal	$= R1\,000 - R1\,000$	$R0$

Completing this table for a three year period does not take too long. However, if Kate needed to make annual payments for 20 years, then the calculation becomes very repetitive and time-consuming. Therefore, we need a more efficient method for performing these calculations.

4.1 Deriving the formula

In the example above, Kate needed to deposit:

$$\begin{aligned}R2\,486.55 &= R909.09 + R826.45 + R751.31 \\ &= 1\,000(1+0.1)^{-1} + 1\,000(1+0.1)^{-2} + 1\,000(1+0.1)^{-3}\end{aligned}$$

We notice that this is a geometric series with a constant ratio $r = (1 + 0,1)^{-1}$.
Using the formula for the sum of a geometric series:

$$
\begin{aligned}
a &= 1\ 000(1 + 0.1)^{-1} \\
r &= (1 + 0.1)^{-1} \\
n &= 3 \\
S_n &= \frac{a(1 - r^n)}{1 - r} \text{ (for } r < 1\text{)} \\
&= \frac{1\ 000(1 + 0.1)^{-1}[1 - ((1 + 0.1)^{-1})^3]}{1 - (1 + 0.1)^{-1}} \\
&= \frac{1\ 000[1 - (1 + 0,1)^{-3}]}{(1 + 0.1)[1 - (1 + 0.1)^{-1}]} \\
&\frac{1\ 000[1 - (1 + 0.1)^{-3}]}{(1 + 0.1) - 1} \\
&= \frac{1\ 000[1 - (1 + 0.1)^{-3}]}{0.1} \\
&= 2\ 486.85
\end{aligned}
$$

We can therefore use the formula for the sum of a geometric series to derive a formula for the present value (P) of a series of (n) regular payments of an amount (x) which are subject to an interest rate (i):

$$
\begin{aligned}
a &= x(1 + i)^{-1} \\
r &= (1 + i)^{-1} \\
S_n &= \frac{a(1 - r^n)}{1 - r} \text{ (for r < 1)} \\
\therefore P &= \frac{x(1 + i)^{-1}[1 - ((1 + i)^{-1})^n]}{1 - (1 + i)^{-1}} \\
&= \frac{x[1 - (1 + i)^{-n}]}{(1 + i)[1 - (1 + i)^{-1}]} \\
&= \frac{x[1 - (1 + i)^{-n}]}{1 + i - 1} \\
&= \frac{x[1 - (1 + i)^{-n}]}{i}
\end{aligned}
$$

Present value of a series of payments:

$$P = \frac{x[1 - (1 + i)^{-n}]}{i}$$

If we are given the present value of a series of payments, we can calculate the value of the payments by making x the subject of the above formula.

Payment amount:

$$x = \frac{P \times i}{[1 - (1 + i)^{-n}]}$$

5 ANALYSING INVESTMENTS AND LOAN OPTIONS

In the next worked example we consider the effects of the duration of the repayment period on the total amount repaid (the amount borrowed plus the accrued interest) for a loan.

WORKED EXAMPLE 11: REPAYMENT PERIODS

QUESTION

David and Julie take out a home loan of R $2,6$ million with an interest rate of 10% per annum compounded monthly.

1. Calculate the monthly repayments for a repayment period of 30 years.
2. Calculate the interest paid on the loan at the end of the 30 year period.
3. Determine the monthly repayments for a repayment period of 20 years.
4. Determine the interest paid on the loan at the end of the 20 year period.
5. What is the difference in the monthly repayment amounts?
6. Comment on the difference in the interest paid for the two different time periods.

SOLUTION

Step 1: Consider a 30 year repayment period on the loan

$$\begin{aligned} x &= \frac{P \times i}{[1 - (1+i)^{-n}]} \\ P &= \text{R}2\ 600\ 000 \\ i &= \frac{0,1}{12} \\ n &= 30 \times 12 = 360 \\ x &= \frac{2\ 600\ 000 \times \frac{0,1}{12}}{\left[1 - \left(1 + \frac{0,1}{12}\right)^{-360}\right]} \\ &= R22\ 816,86 \end{aligned}$$

Therefore, the monthly repayment is R$22\ 816,86$ for a 30 year period.

WORKED EXAMPLE 11: REPAYMENT PERIODS (Continues...)

At the end of the 30 years, David and Julie will have paid a total amount of:

$$= 30 \times 12 \times \text{R}22\ 816,86$$
$$= R8\ 214\ 069,60$$

The total amount of interest on the loan:

$$\text{Interest} = \text{total amount paid – loan amount} = R8\ 214\ 069,60 \text{ – R}2\ 600\ 000$$
$$= R5\ 614\ 069,60$$

We notice that the interest on the loan is more than double the amount borrowed.

Step 2: Consider a 20 year repayment period on the loan

$$x = \frac{P \times i}{[1-(1+i)^{-n}]}$$
$$P = R2\ 600\ 000$$
$$i = \frac{0,1}{12}$$
$$n = 20\times$$
$$12 = 240$$
$$x = \frac{2\ 600\ 000 \times \frac{0,1}{12}}{\left[1-\left(1+\frac{0,1}{12}\right)^{-240}\right]}$$
$$= R25\ 090,56$$

Therefore, the monthly repayment is R$25090,56$ for a 20 year period. At the end of the 20 years, David and Julie will have paid a total amount of:

$$= 20×12×R25\ 090,$$
$$= R6\ 021\ 734,40$$

WORKED EXAMPLE 11: REPAYMENT PERIODS (Continues...)

The total amount of interest on the loan:

$$\begin{aligned}\text{Interest} &= \text{total amount paid} - \text{loan amount} \\ &= R6\ 021\ 734,40 - R2\ 600\ 000 \\ &= R3\ 421\ 734,40\end{aligned}$$

We notice that the interest on the loan is about $1,3$ times the borrowed amount.

Step 3: Consider the difference in the repayment and interest amounts

$$\begin{aligned}\text{Difference in repayments} &= R25\ 090,56 - R22\ 816,86 \\ &= R2\ 273,70\end{aligned}$$

It is also very interesting to look at the difference in the total interest paid:

$$\begin{aligned}\text{Difference in interest} &= R5\ 614\ 069,60 - R3\ 421\ 734,40 \\ &= R2\ 192\ 335,20\end{aligned}$$

Therefore, by paying an extra R$2\ 273,70$ each month over a shorter repayment period, David and Julie could save more than R2 million on the repayment of their home loan.

When considering taking out a loan, it is advisable to investigate and compare a few options offered by financial institutions. It is very important to make informed decisions regarding personal finances and to make sure that the monthly repayment amount is serviceable (payable). A credit rating is an estimate of a person's ability to fulfill financial commitments based on their previous payment history. Defaulting on a loan can affect a person's credit rating and their chances of taking out another loan in the future.

WORKED EXAMPLE 12: ANALYSING INVESTMENT OPPORTUNITIES

QUESTION

Marlene wants to start saving for a deposit on a house. She can afford to invest between R400 and R600 each month and gets information from four different investment firms. Each firm quotes a different interest rate and a prescribed monthly installment amount. She plans to buy a house in 7 years' time. Calculate which company offers the best investment opportunity for Marlene.

	Interest rate (compounded monthly)	Monthly payment
TBS Investments	$13,5\%$ p.a.	$R450$
Taylor Anderson	13% p.a.	$R555$
PHK	$12,5\%$ p.a.	$R575$
Simfords Consulting	11% p.a.	$R600$

SOLUTION

Step 1: Consider the different investment options

To compare the different investment options, we need to calculate the following for each option at the end of the seven year period:

- The future value of the monthly payments.
- The total amount paid into the investment fund.
- The total interest earned.

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

TBS Investments:

$$\begin{aligned} F &= \frac{450\left[\left(1+\frac{0{,}135}{12}\right)^{84} - 1\right]}{\frac{0{,}135}{12}} \\ &= \text{R}62\ 370{,}99 \\ \text{Total amount } (T): &= 7 \times 12 \times \text{R}450 \\ &= \text{R}37\ 800 \\ \text{Total interest } (I): &= \text{R}62\ 370{,}99 - \text{R}37\ 800 \\ &= \text{R}24\ 570{,}99 \end{aligned}$$

Taylor Anderson:

$$\begin{aligned} F &= \frac{555\left[\left(1+\frac{0{,}13}{12}\right)^{84} - 1\right]}{\frac{0{,}13}{12}} \\ &= \text{R}75\ 421{,}65 \end{aligned}$$

WORKED EXAMPLE 12: ANALYSING INVESTMENT OPPORTUNITIES (Continues...)

$$\begin{aligned}\text{Total amount } (T) &:= 7 \times 12 \times \text{R}555 \\ &= \text{R}46\,620 \\ \text{Total interest } (I) &:= \text{R}75\,421,65 - \text{R}46\,620 \\ &= \text{R}28\,801,65\end{aligned}$$

PHK:

$$\begin{aligned}F &= \frac{575\left[\left(1+\frac{0,125}{12}\right)^{84}-1\right]}{\frac{0,125}{12}} \\ &= \text{R}76\,619,96 \\ \text{Total amount } (T) &:= 7 \times 12 \times \text{R}575 \\ &= \text{R}48\,300 \\ \text{Total interest } (I) &:= \text{R}76\,619,96 - \text{R}48\,300 \\ &= \text{R}28\,319,96\end{aligned}$$

Simfords Consulting:

$$\begin{aligned}F &= \frac{600\left[\left(1+\frac{0,11}{12}\right)^{84}-1\right]}{\frac{0,11}{12}} \\ &= \text{R}75\,416,96 \\ \text{Total amount } (T) &:= 7 \times 12 \times \text{R}600 \\ &= \text{R}50\,400 \\ \text{Total interest } (I) &:= \text{R}75\,416,96 - \text{R}50\,400 \\ &= \text{R}25\,016,96\end{aligned}$$

Step 2: Draw a table of the results to compare the answers

	F	T	I
TBS Investments	R62 370,99	R37 800	R24 570,99
Taylor Anderson	R75 421,65	R46 620	R28 801,65
PHK	R76 619,96	R48 300	R28 319,96
Simfords Consulting	R75 416,96	R50 400	R25 016,96

Step 3: Make a conclusion

An investment with PHK would provide Marlene with the highest deposit (R76 619,96) for her house at the end of the 7 year period. However, we notice that an investment with Taylor Anderson would earn the highest amount of interest (R28 801,65) and is therefore the better investment option.

WORKED EXAMPLE 13: ANALYSING LOAN OPTIONS

QUESTION

William wants to take out a loan of R$750\ 000$, so he approaches three different banks. He plans to start repaying the loan immediately and he calculates that he can afford a monthly repayment amount between R$5\ 500$ and R$7\ 000$.

Calculate which of the three options would be best for William.

- West Bank offers a repayment period of 30 years and an interest rate of prime compounded monthly.
- AcuBank offers a repayment period of 20 years and an interest rate of prime $+0,5$% compounded monthly.
- FinTrust Bank offers a repayment period of 15 years and an interest rate of prime $+2$% compounded monthly.

SOLUTION

Step 1: Consider the different loan options

To compare the different loan options, we need to calculate the following for each option:

- The monthly payment amount.
- The total amount paid to repay the loan.
- The amount of interest on the loan.

$$x = \frac{P \times i}{[1 - (1+i)^{-n}]}$$

West Bank:

$$\begin{aligned} x &= \frac{750\ 000 \times \frac{0,085}{12}}{\left[1 - \left(1 + \frac{0,085}{12}\right)^{-360}\right]} \\ &= \text{R}5\ 766,85 \\ \text{Total amount } (T): &= 30 \times 12 \times \text{R}5\ 766,85 \\ &= \text{R}2\ 076\ 066 \\ \text{Total interest } (I): &= \text{R}2\ 076\ 066 - \text{R}750\ 000 \\ &= \text{R}1\ 326\ 066 \end{aligned}$$

WORKED EXAMPLE 13: ANALYSING LOAN OPTIONS (Continues...)

AcuBank:

$$
\begin{aligned}
x &= \frac{750\ 000 \times \frac{0{,}09}{12}}{\left[1-\left(1+\frac{0{,}09}{12}\right)^{-240}\right]} \\
&= \text{R}6\ 747{,}94 \\
\text{Total amount } (T): &= 20 \times 12 \times \text{R}6\ 747{,}94 \\
&= \text{R}1\ 619\ 505{,}60 \\
\text{Total interest } (I): &= \text{R}1\ 619\ 505{,}60 - \text{R}750\ 000 \\
&= \text{R}869\ 505{,}60
\end{aligned}
$$

FinTrust Bank:

$$
\begin{aligned}
x &= \frac{750\ 000 \times \frac{0{,}105}{12}}{\left[1-\left(1+\frac{0{,}105}{12}\right)^{-180}\right]} \\
&= \text{R}8\ 290{,}49 \\
\text{Total amount } (T): &= 15 \times 12 \times \text{R}8\ 290{,}49 \\
&= \text{R}1\ 492\ 288{,}20 \\
\text{Total interest } (I): &= \text{R}1\ 492\ 288{,}20 - \text{R}750\ 000 \\
&= \text{R}742\ 288{,}20
\end{aligned}
$$

Step 2: Draw a table of the results to compare the answers

	x	T	I
West Bank	R5 766, 85	R2 076 066, 00	R1 326 066, 00
AcuBank	R6 747, 94	R1 619 505, 60	R869 505, 60
FinTrust Bank	R8 290, 49	R1 492 288, 20	R742 288, 20

Step 3: Make a conclusion

A loan from FinTrust Bank would accumulate the lowest amount of interest but the monthly repayment amounts are not within William's budget. Although West Bank offers the lowest interest rate and monthly repayment amount, the interest earned on the loan is very high as a result of the longer repayment period. If we assume that William must repay the loan over the given time periods, then AcuBank offers the best option.

However, we know that William can afford to pay more than R5 766, 85 per month, and if the bank allows him to pay back the loan earlier, he should consider taking out a loan with West Bank and take advantage of the lower interest rate.

6 SUMMARY

- Always keep the rate of interest per time unit and the time period in the same units.
- Simple interest: $A = P(1 + in)$
- Compound interest: $A = P(1 + i)^n$
- Simple depreciation: $A = P(1 - in)$
- Compound depreciation: $A = P(1 - i)^n$
- Nominal and effective annual interest rates: $1 + i = \left(1 + \frac{i^m}{m}\right)^m$
- **Future value of payments:**

$$F = \frac{x[(1 + i)^n - 1]}{i}$$

  Payment amount:

$$x = \frac{F \times i}{[(1 + i)^n - 1]}$$

- **Present value of a series of payments:**

$$P = \frac{x[1 - (1 + i)^{-n}]}{i}$$

  Payment amount:

$$x = \frac{P \times i}{[1 - (1 + i)^{-n}]}$$

7 EXERCISES

7.1 Exercise 1

1. Nzuzo invests R $80\ 000$ at an interest rate of $7,5$% per annum compounded yearly.
   How long will it take for his investment to grow to R $100\ 000$?

2. Sally invests R $120\ 000$ at an interest rate of 12% per annum compounded quarterly.
   How long will it take for her investment to double?

3. When Banele was still in high school he deposited R $2\ 250$ into a savings account with an interest rate of $6,99$% per annum compounded yearly. How long ago did Banele open the account if the balance is now R $2\ 882,53$? Write the answer as a combination of years and months.

4. The annual rate of depreciation of a vehicle is 15% . A new vehicle costs R $122\ 000$. After how many years will the vehicle be worth less than R $40\ 000$?

5. Some time ago, a man opened a savings account at KMT South Bank and deposited an amount of R $2\ 100$. The balance of his account is now R $3\ 160,59$. If the account gets$8,52$% compound interest p.a., determine how many years ago the man made the deposit.

6. Mr. and Mrs. Dlamini want to save money for their son's university fees. They deposit R $7\ 000$ in a savings account with a fixed interest rate of $6,5$% per year compounded annually. How long will it take for this deposit to double in value?

7. A university lecturer retires at the age of 60 . She has saved R $300\ 000$ over the years.

   7.1 She decides not to let her savings decrease at a rate faster than 15% per year. How old will she be when the value of her savings is less than R $50\ 000$?

   7.2 If she doesn't use her savings and invests all her money in an investment account that earns a fixed interest rate of $5,95$% per annum, how long will it take for her investment to grow to R $390\ 000$?

8. Simosethu puts R 450 into a bank account at the Bank of Upington. Simosethu's account pays interest at a rate of $7,11$% p.a. compounded monthly. After how many years will the bank account have a balance of R $619,09$?

7.2 Exercise 2

1. Shelly decides to start saving money for her son's future. At the end of each month she deposits R 500 into an account at Durban Trust Bank, which earns an interest rate of $5,96\%$ per annum compounded quarterly.

    1. 1 Determine the balance of Shelly's account after 35 years.
    1. 2 How much money did Shelly deposit into her account over the 35 year period?
    1. 3 Calculate how much interest she earned over the 35 year period.

2. Gerald wants to buy a new guitar worth R $7\ 400$ in a year's time. How much must he deposit at the end of each month into his savings account, which earns a interest rate of $9,5\% p.a.$ compounded monthly?

3. A young woman named Grace has just started a new job, and wants to save money for the future. She decides to deposit R $1\ 100$ into a savings account every month.Her money goes into an account at First Mutual Bank, and the account earns $8,9\%$ interest p.a. compounded every month.

    3. 1 How much money will Grace have in her account after 29 years?
    3. 2 How much money did Grace deposit into her account by the end of the 29 year period?

4. Ruth decides to save for her retirement so she opens a savings account and immediately deposits R 450 into the account. Her savings account earns 12% per annum compounded monthly. She then deposits R 450 at the end of each month for 35 years. What is the value of her retirement savings at the end of the 35 year period?

5. Musina MoneyLenders offer a savings account with an interest rate of $6,13\%$ p.a. compounded monthly. Monique wants to save money so that she can buy a house when she retires. She decides to open an account and make regular monthly deposits. Her goal is to end up with R $750\ 000$ in her account after 35 years.

    5. 1 How much must Monique deposit into her account each month in order to reach her goal?
    5. 2 How much money, to the nearest rand, did Monique deposit into her account by the end of the 35 year period?

6. Lerato plans to buy a car in five and a half years' time. She has saved R $30\ 000$ in a separate investment account which earns 13% per annum compound interest. If she doesn't want to spend more than R $160\ 000$ on a vehicle and her savings account earns an interest rate of 11% p.a. compounded monthly, how much must she deposit into her savings account each month?

7. Every Monday Harold puts R30 into a savings account at the King bank, which accrues interest of $6,92\%$ p.a. compounded weekly.

    7. 1 How long will it take Harold's account to reach a balance of R $4\ 397,53$? Give the answer as a number of years and days to the nearest integer.
    7. 2 How much interest will Harold receive from the bank during the period of his investment?

7.3 Exercise 3

1. Mfethu owns his own delivery business and he will need to replace his truck in 6 years' time. Mfethu deposits R $3\ 100$ into a sinking fund each month, which earns $5,3$% interest p.a. compounded monthly.

   1. 1 How much money will be in the fund in 6 years' time, when Mfethu wants to buy the new truck?

   1. 2 If a new truck costs R $285\ 000$ in 6 years' time, will Mfethu have enough money to buy it?

2. Atlantic Transport Company buys a van for R $265\ 000$. The value of the van depreciates on a reducing-balance basis at 17% per annum. The company plans to replace this van in five years' time and they expect the price of a new van to increase annually by 12% .

   2. 1 Calculate the book value of the van in five years' time.

   2. 2 Determine the amount of money needed in the sinking fund for the company to be able to afford a new van in five years' time.

   2. 3 Calculate the required monthly deposits if the sinking fund earns an interest rate of 11% per annum compounded monthly.

3. Tonya owns Freeman Travel Company and she will need to replace her computer in 7 years' time. Tonya creates a sinking fund so that she will be able to afford a new computer, which will cost R $8\ 450$. The sinking fund earns interest at a rate of $7,67$% p.a. compounded each quarter.

   3. 1 How much money must Tonya save quarterly so that there will be enough money in the account to buy the new computer?

   3. 2 How much interest (to the nearest rand) does the bank pay into the account by the end of the 7 years period?

4. Gemima owns SupaClean Laundry. She just bought new washing machines at a cost of R $55\ 000$. The washing machines depreciate at a rate of 15% per annum on the reducing balance method. The washing machines will need to be replaced in 5 years time.

   4. 1 Calculate the scrap value of the washing machine in 5 years' time.

   4. 2 If the inflation rate of washing machines is expected to be $4,7$% per annum for the next 5 years, calculate the replacement cost of the washing machines in 5 years' time.

   4. 3 Determine the amount of money required in the sinking fund in 5 years' time in order to replace the old washing machines. Assume that the old washing machines will be sold at scrap value to contribute toward the new washing machines.

   4. 4 Calculate the required monthly deposit amount into the sinking fund for Gemima to be able to afford the new washing machines in 5 years' time. The company earns an interest rate of $8,7$% per annum compounded monthly on the sinking fund.

4. 5 At the end of every year Gemima needs to do maintenance on the washing machines. The cost of the maintenance is R $4\ 500$ per year for all the washing machines. How much money will be short in the sinking fund if Gemima takes the money from the sinking fund in order to do the maintenance?

4. 6 Calculate the required monthly deposit that will be needed if Gemima wants to buy the new washing machines in 5 years as well as do the maintenance on the machines?

7.4 Exercise 4

1. A property costs R $1\ 800\ 000$. Calculate the monthly repayments if the interest rate is 14% p.a. compounded monthly and the loan must be paid off in 20 years' time.

2. A loan of R $4\ 200$ is to be returned in two equal annual installments. If the rate of interest is 10% compounded annually, calculate the amount of each installment.

3. Stefan and Marna want to buy a flat that costs R $1,2$ million.
   Their parents offer to put down a 20% payment towards the cost of the house.
   They need to get a mortgage for the balance.
   What is the monthly repayment amount if the term of the home loan is 30 years and the interest is $7,5\%$ p.a. compounded monthly?

4. Ziyanda arranges a bond for R $17\ 000$ from Langa Bank.
   If the bank charges $16,0\%$ p.a. compounded monthly.

   4. 1 Determine Ziyanda's monthly repayment if she is to pay back the bond over 9 years.

   4. 2 What is the total cost of the bond?

5. Dullstroom Bank offers personal loans at an interest rate of $15,63\%$ p.a. compounded twice a year. Lubabale borrows R $3\ 000$ and must pay R $334,93$ every six months until the loan is fully repaid.

   5. 1 How long will it take Lubabale to repay the loan?

   5. 2 How much interest will Lubabale pay for this loan?

6. Likengkeng has just started a new job and wants to buy a car that costs R $232\ 000$. She visits the Soweto Savings Bank, where she can arrange a loan with an interest rate of $15,7\%$ p.a. compounded monthly. Likengkeng has enough money saved to pay a deposit of R $50\ 000$. She arranges a loan for the balance of the payment, which is to be paid over a period of 6 years.

   6. 1 What is Likengkeng's monthly repayment on her loan?

   6. 2 How much will the car cost Likengkeng?

7. Anathi is a wheat farmer and she needs to buy a new holding tank which costs R $219\ 450$. She bought her old tank 14 years ago for R $196\ 000$. The value of the old grain tank has depreciated at a rate of $12,1\%$ per year on a reducing balance,and she plans to trade it in for its current value. Anathi will then need

to arrange a loan for the balance of the cost of the new grain tank. Orsmond bank offers loans with an interest rate of $9,71\%$ p.a. compounded monthly for any loan up to R170 000 and $9,31\%$ p.a. compounded monthly for a loan above that amount. The loan agreement allows Anathi a grace period for the first six months (no payments are made) and it states that the loan must be repaid over 30 years.

7. 1 Determine the monthly repayment amount.

7. 2 What is the total amount of interest Anathi will pay for the loan?

7. 3 How much money would Anathi have saved if she did not take the six month grace period?

7.5 Exercise 5

1. Jason takes out a home loan over 25 years to buy a house that costs R$2\ 500\ 000$. If his monthly installment is R$23\ 604,54$ with interest charged at $10,5\%$ p.a compounded monthly.
   Calculate the outstanding balance immediately after the 245^{th} payment was made.

2. Joan took out a loan of R$94\ 000$ with interest at 15% p.a, compounded monthly. Payments of R$4\ 100$ will be made on the first day of each month.

   2. 1 How many installments of R$4\ 100$ must she pay?

   2. 2 Calculate the final payment, to the nearest rand, Joan has to pay to settle the loan.

3. A bank granted Selo a loan of $R240\ 000$ at an interest rate of 17% p.a, compounded monthly, to buy a car. Selo agreed to repay the loan in monthly installments of R$8\ 600$.

   3. 1 How many installments of R$8\ 600$ must he pay?

   3. 2 Calculate the final installment.

4. Cindy takes out a loan of R$120\ 000$ for home improvements. The loan is taken over four years at an interest rate of 12% per annum compounded monthly.

   4. 1 Calculate the monthly payments if the first payment is made one month after the loan is granted.

   4. 2 Calculate the outstanding balance after the 20^{th} payment has been made.

5. A mortgage of R$190\ 000$ is required to purchase a house. The mortgage will be repaid with a set monthly payment of R$1\ 400$ over 25 years at 8% compounded monthly. Calculate the outstanding balance after the 220^{th} payment has been made.

7.6 Exercise 6

1. Cokisa is 31 years old and starting to plan for her future.
   She has been thinking about her retirement and wants to open an annuity so that she will have money when she retires.
   Her intention is to retire when she is 65 years old.
   Cokisa visits the Trader's Bank of Tembisa and learns that there are two investment options from which she can choose:
   •Option A : $7,76\%$ p.a. compounded once every four months
   •Option B : $7,78\%$ p.a. compounded half-yearly

   1. 1 Which is the better investment option for Cokisa if the amount she will deposit will always be the same?

   1. 2 Cokisa opens an account and starts saving R $4\ 000$ every four months.
   How much money (to the nearest rand) will she have saved when she reaches her planned retirement?

2. Phoebe wants to take out a home loan of R $1,6$ million.
   She approaches three different banks for their loan options:
   •Bank A offers a repayment period of 30 years and
   an interest rate of 12% per annum compounded monthly.
   •Bank B offers a repayment period of 20 years and
   an interest rate of 14% per annum compounded monthly.
   •Bank C offers a repayment period of 30 years and
   an interest rate of 14% per annum compounded monthly.
   If Phoebe intends to start her monthly repayments immediately,
   calculate which of the three options would be best for her.

3. Siyanda has a choice between two investment opportunities.
   Option A : R$1\ 500$ investment on a monthly basis for7 years at an interest rate of 15% per annum compounded monthly.
   Option B : R$1\ 500$ investment on a monthly basis for 10 years at an interest rate of 13% per annum compounded monthly.

8 ANSWERS TO EXERCISES:

8.1 Exercise 1

1. It will take just over $3,09$ years
2. It will take just over 5 years and 10 months.
3. Banele deposited the money into the account 3 years and 8 months ago.
4. The vehicle will be worth less than R$40\ 000$ after about $6,86$ years.
5. The man made the deposit 5 years ago.
6. It will take 11 years for their deposit to double in value.

7. 1 If she manages her money carefully, she will be 71 years or older.

7. 2 It will take less than 5 years.

8. The money has been in the Simosethu's account for $4,5$ years.

8.2 Exercise 2

1. 1 R $232\ 539,41$

1. 2 R $210\ 000$

1. 3 R $22\ 539,41$

2. Gerald must deposit R $590,27$ each month so that he can afford his guitar.

3. 1 R $1\ 792400.11$

3. 2 After 29 years, Grace deposited a total of R $382\ 800$ into her account.

4. Ruth will have saved R $2\ 923\ 321,08$ for her retirement.

5. 1 In order to save R $750\ 000$ in 35 years, Monique will need to save R $510,85$ in her account every month.

5. 2 After 35 years, Monique deposited a total of R $214\ 557$ into her account.

6. Lerato must deposit R $1\ 123,28$ each month into her savings account.

7. 1 Harold's investment takes 2 years and 211 days to reach the final value of R $4\ 397,53$.

7. 2 The total amount of interest paid by the bank:
R $4\ 397,53$–R $4\ 020,00 =$ R $377,53$.

8.3 Exercise 3

1. 1 After 6 years, Mfethu will have R $262\ 094,55$ in his sinking fund.

1. 2 No, Mfethu does not have enough money in his account:
R $285\ 000$–R $262\ 094,55 =$ R $22\ 905,45$

2. 1 R $104\ 384,58$

2. 2 R $362\ 635,97$

2. 3 The company must deposit R $4\ 560,42$ each month.

3. 1 Tonya must deposit R $230,80$ into the sinking fund quarterly.

3. 2 To the nearest rand, the bank paid R $1\ 988$ into the account.

4. 1 R$24\ 403,79$

4. 2 R$69\ 198,41$

4. 3 R$44\ 794,62$

4. 4 R$598,59$

4. 5 R$19\ 349,48$

4. 6 R$938,62$

8.4 Exercise 4

1. R $22\ 383,37$

2. R $2\ 420,00$

3. R $6\ 712,46$

4. 1 Ziyanda must pay R $297,93$ each month.

4. 2 Ziyanda paid the bank a total of R $32\ 176,44$.

5. 1 It will take 8 years.

5. 2 R $2\ 358,88$

6. 1 Likengkeng must pay R $3\ 917,91$ each month.

6. 2 R $332\ 089,87$

7. 1 Anathi must pay R $1\ 627,05$ each month.

7. 2 The total amount of interest Anathi paid is R $388\ 743,83$.

7. 3 R $18\ 530,10$

8.5 Exercise 5

1. R$1\ 026\ 985,22$

2. 1 $n = 27,18 \dots \approx 27$ full payments

2. 2 R$757,81$

3. 1 $n = 35,76 \dots \approx 35$ full payments

3. 2 R$6\ 582,63$

4. 1 R$3\ 160,06$

4. 2 R$76\ 841,42$

5. R$86\ 586,05$

8.6 Exercise 6

1. 1 Option A is the better one.

1. 2 Cokisa will have R $1\ 937\ 512,76$ for her retirement.

2. Phoebe should consider taking out a loan with Bank B, as this has the lowest total repayment amount.

3. Option B