CHAPTER 4

Trigonometry

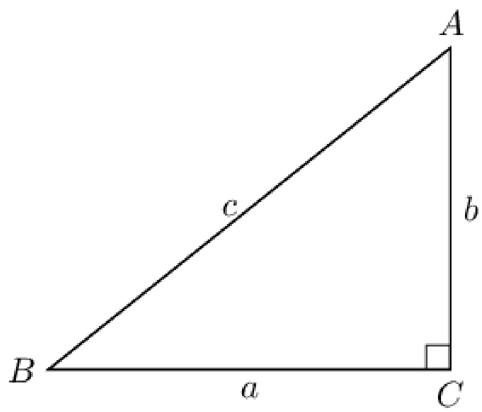
CONTENTS

April 20, 2021

1 REVISION

Trigonometric ratios

We defined the basic trigonometric ratios using the lengths of the sides of a right-angled triangle.



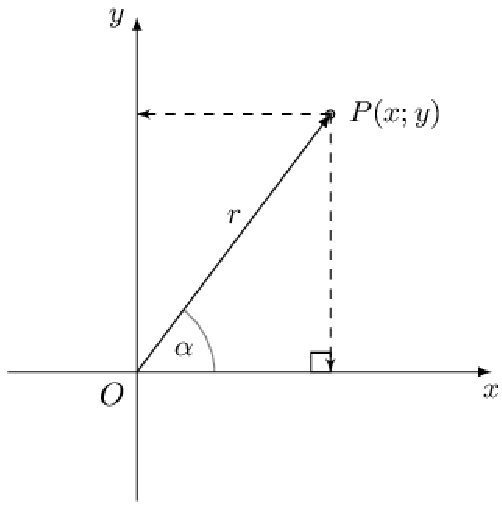
$$\sin \hat{A} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \qquad \sin \hat{B} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\cos \hat{A} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} \qquad \cos \hat{B} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\tan \hat{A} = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} \qquad \tan \hat{B} = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

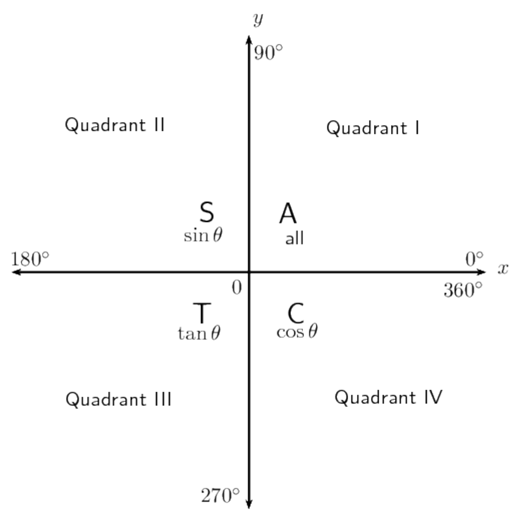
Trigonometric ratios in the Cartesian plane

We also defined the trigonometric ratios with respect to any point in the Cartesian plane in terms of x, y and r. Using the theorem of Pythagoras, $r^2 = x^2 + y^2$.



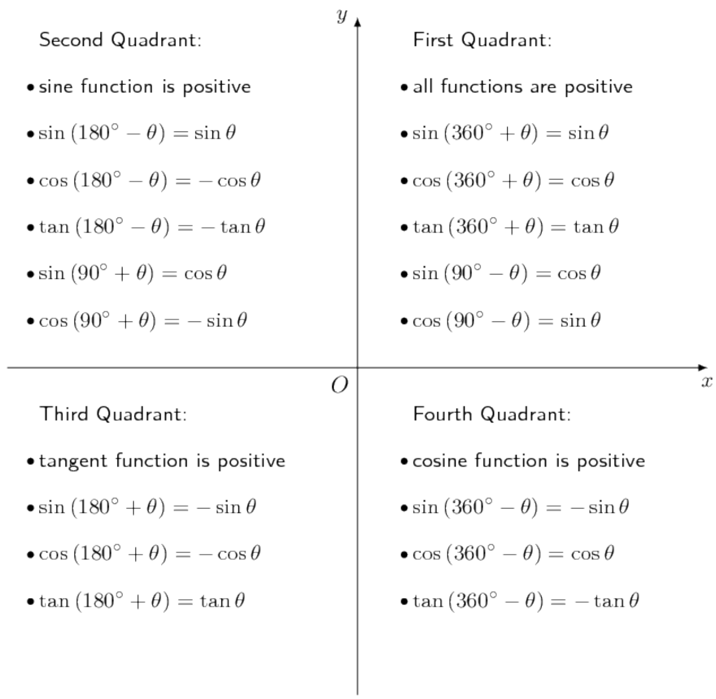
CAST diagram

The sign of a trigonometric ratio depends on the signs of x and y:



Reduction formulae and co-functions:

1. The reduction formulae hold for any angle θ. For convenience, we assume θ is an acute angle ($0° < \theta < 90°$).
2. When determining function values of ($180° \pm \theta$), ($360° \pm \theta$) and ($-\theta$) the function does not change.
3. When determining function values of ($90° \pm \theta$) and ($\theta \pm 90°$) the function changes to its co-function.

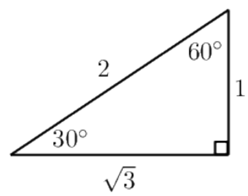


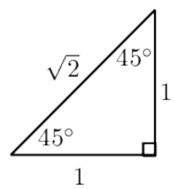
Negative angles

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

Special angle triangles

These values are useful when we need to solve a problem involving trigonometric functions without using a calculator. Remember that the lengths of the sides of a right-angled triangle obey the theorem of Pythagoras.





θ	0°	30°	45°	60°	90°
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

Trigonometric identities

Quotient identity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad (\cos\theta \neq 0)$$

Square identity:

$$\sin^2\theta + \cos^2\theta = 1$$

It also follows that:

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta$$

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$
$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

All these relationships and identities are very useful for simplifying trigonometric expressions.

WORKED EXAMPLE 1: REVISION

QUESTION

Determine the value of the expression, without using a calculator:

$$\frac{\cos 420^\circ - \sin 225^\circ \cos(-45^\circ)}{\tan 315^\circ}$$

SOLUTION

Step 1: Use reduction formulae to express each trigonometric ratio in terms of an acute angle

$$\begin{aligned}
&\frac{\cos 420^\circ - \sin 225^\circ \cos(-45^\circ)}{\tan 315^\circ}\\
&= \frac{\cos(360^\circ + 60^\circ) - \sin(180^\circ + 45^\circ)\cos(-45^\circ)}{\tan(360^\circ - 45^\circ)}\\
&= \frac{\cos 60^\circ - (-\sin 45^\circ)(\cos 45^\circ)}{-\tan 45^\circ}\\
&= \frac{\cos 60^\circ + \sin 45^\circ \cos 45^\circ}{-\tan 45^\circ}
\end{aligned}$$

Now use special angles to evaluate the simplified expression:

$$\begin{aligned}
&= \frac{\cos 60^\circ + \sin 45^\circ \cos 45^\circ}{-\tan 45^\circ}\\
&= \frac{\frac{1}{2} + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}{-1}\\
&= -\left(\frac{1}{2} + \frac{1}{2}\right)\\
&= -1
\end{aligned}$$

WORKED EXAMPLE 2: REVISION

QUESTION

Prove:

$$\sin^2\alpha - (\tan\alpha - \cos\alpha)(\tan\alpha + \cos\alpha) = \frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}$$

State restrictions where applicable.

SOLUTION

Step 1 : Use trigonometric identities to simplify each side separately.

Simplify the left-hand side of the identity

$$\begin{aligned} \text{LHS} &= \sin^2\alpha - (\tan\alpha - \cos\alpha)(\tan\alpha + \cos\alpha) \\ &= \sin^2\alpha - (\tan^2\alpha - \cos^2\alpha) \\ &= \sin^2\alpha - \tan^2\alpha + \cos^2\alpha \\ &= \left(\sin^2\alpha + \cos^2\alpha\right) - \tan^2\alpha \\ &= 1 - \tan^2\alpha \end{aligned}$$

Simplify the right-hand side of the identity so that it equals the left-hand side:

$$\begin{aligned} \text{RHS} &= \frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha} \\ &= \frac{\cos^2\alpha}{\cos^2\alpha} - \frac{\sin^2\alpha}{\cos^2\alpha} \\ &= 1 - \tan^2\alpha \end{aligned}$$

$$\therefore \text{LHS} = \text{RHS}$$

Alternative method: we could also have started with the left-hand side of the identity and substituted $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and simplified to get the right-hand side.

Restrictions

We need to determine the values of α for which any of the terms in the identity will be undefined:

$$\begin{aligned} \cos^2\alpha &= 0 \\ \therefore \cos\alpha &= 0 \\ \therefore \alpha &= 90^\circ \text{ or } 270^\circ \end{aligned}$$

We must also consider the values of α for which $\tan\alpha$ is undefined. Therefore, the identity is undefined for $\alpha = 90^\circ + k \cdot 180^\circ$.

Useful tips:

- It is sometimes useful to write $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$;
- Never write a trigonometric ratio without an angle. For example, $\tan = \frac{\sin}{\cos}$ has no meaning.
- For proving identities, only simplify one side of the identity at a time.
- As seen in the worked example above, sometimes both sides of the identity need to be simplified.
- Remember to write down restrictions:
  - the values for which any of the trigonometric ratios are not defined;
  - the values of the variable which make any of the denominators in the identity equal to zero.

2 COMPOUND ANGLE IDENTITIES

2.1 Derivation of $\cos(\alpha - \beta)$

INVESTIGATION

Compound angles

Danny is studying for a trigonometry test and completes the following question:

Question:

Evaluate the following:

$\cos(180° - 120°)$

Danny's solution:

$$\begin{aligned} \cos(180° - 120°) &= \cos 180° - \cos 120° && \text{(line 1)} \\ &= -1 - \cos(90° + 30°) && \text{(line 2)} \\ &= -1 + \sin 30° && \text{(line 3)} \\ &= -1 + \tfrac{1}{2} && \text{(line 4)} \\ &= -\tfrac{1}{2} && \text{(line 5)} \end{aligned}$$

1. Consider Danny's solution and determine why it is incorrect.
2. Use a calculator to check that Danny's answer is wrong.
3. Describe in words the mistake(s) in his solution.
4. Is the following statement true or false? "A trigonometric ratio can be distributed to the angles that lie within the brackets."

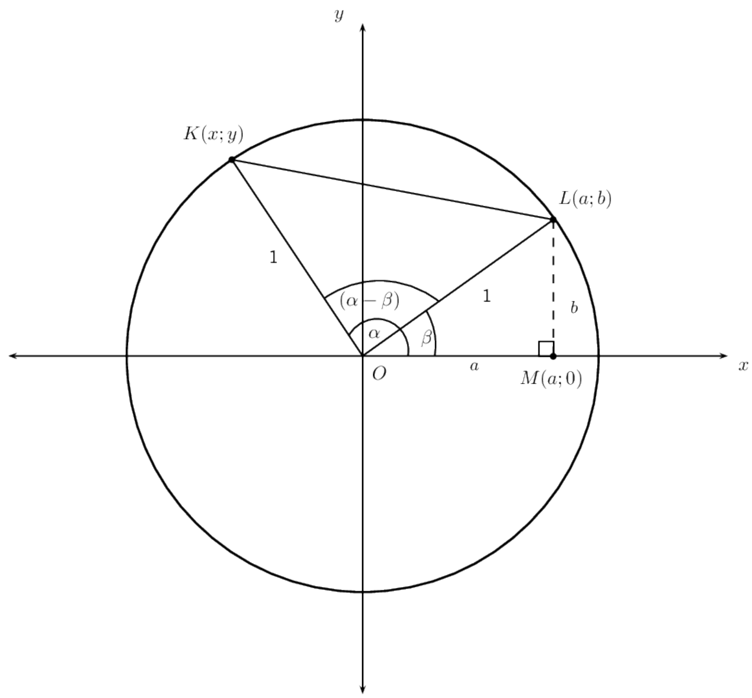
From the investigation above, we know that $\cos(\alpha - \beta) \neq \cos\alpha - \cos\beta$. It is wrong to apply the distributive law to the trigonometric ratios of compound angles.

$$\text{Distance formula:} \quad d_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$
$$\text{Cosine rule:} \quad a^2 = b^2 + c^2 - 2bc \cdot \cos\hat{A}$$

Using the distance formula and the cosine rule, we can derive the following identity for compound angles:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Consider the unit circle $(r = 1)$ below. The two points $L\,(a;b)$ and $K\,(x;y)$ are shown on the circle.



We can express the coordinates of L and k n terms of the angles θ and β :

$$\text{In } \triangle LOM, \quad \sin\beta = \frac{b}{1}$$
$$\therefore b = \sin\beta$$
$$\cos\beta = \frac{a}{1}$$
$$\therefore a = \cos\beta$$

$$L = (\cos\beta; \sin\beta)$$

$$\text{Similarly, } K = (\cos\alpha; \sin\alpha)$$

We use the distance formula to determine KL^2

$$\begin{aligned} d^2 &= (x_K - x_L)^2 + (y_K - y_L)^2 \\ KL^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\ &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta \\ &= \left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\ &= 1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \end{aligned}$$

Now we determine KL^2 using the cosine rule for $\triangle KOL$:

$$\begin{aligned} KL^2 &= KO^2 + LO^2 - 2 \cdot KO \cdot LO \cdot \cos(\alpha - \beta) \\ &= 1^2 + 1^2 - 2(1)(1)\cos(\alpha - \beta) \\ &= 2 - 2 \cdot \cos(\alpha - \beta) \end{aligned}$$

Equating the two expressions for KL^2 , we have

$$\begin{aligned} 2 - 2 \cdot \cos(\alpha - \beta) &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ 2 \cdot \cos(\alpha - \beta) &= 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ \therefore \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{aligned}$$

WORKED EXAMPLE 3: DERIVATION OF $\cos(\alpha + \beta)$

QUESTION

Derive an expression for $\cos(\alpha + \beta)$ in terms of the trigonometric ratios of α and β.

SOLUTION

Step 1: Use the compound angle formula for $\cos(\alpha - \beta)$

We use the compound angle formula for $\cos(\alpha - \beta)$ and manipulate the sign of β in $\cos(\alpha + \beta)$ so that it can be written as a difference of two angles:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos(\alpha - (-\beta)) \\ \text{And we have shown } \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \therefore \cos[\alpha - (-\beta)] &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\ \therefore \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned}$$

Step 2: Write the final answer

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

WORKED EXAMPLE 4: DERIVATION OF $\sin(\alpha - \beta)$ AND $\sin(\alpha + \beta)$

QUESTION

Derive the expanded formulae for $\sin(\alpha - \beta)$ and $\sin(\alpha + \beta)$ in terms of the trigonometric ratios of α and β.

SOLUTION

Step 1: Use the compound angle formula and co-functions to expand $\sin(\alpha - \beta)$

Using co-functions, we know that $\sin \hat{A} = \cos(90° - \hat{A})$ so we can write $\sin(\alpha + \beta)$ in terms of the cosine function as:

$$\begin{aligned} \sin(\alpha - \beta) &= \cos\left(90° - (\alpha - \beta)\right) \\ &= \cos\left(90° - \alpha + \beta\right) \\ &= \cos\left[(90° - \alpha) + \beta\right] \end{aligned}$$

Apply the compound angle formula:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \therefore \cos\left[(90° - \alpha) + \beta\right] &= \cos(90° - \alpha)\cos\beta - \sin(90° - \alpha)\sin\beta \\ \therefore \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}$$

To derive the formula for $\sin(\alpha + \beta)$, we use the compound formula for $\sin(\alpha - \beta)$ and manipulate the sign of β:

$$\begin{aligned} \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \text{We can write } \sin(\alpha + \beta) &= \sin\left[\alpha - (-\beta)\right] \\ \therefore \sin\left[\alpha - (-\beta)\right] &= \sin\alpha\cos(-\beta) - \cos\alpha\sin(-\beta) \\ \therefore \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

Step 2: Write the final answers

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

2.2 Compound angle formulae

- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$;
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$;
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$;
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Note: We can use the compound angle formulae to expand and simplify compound angles in trigonometric expressions (using the equations from left to right) or we can use the expanded form to determine the trigonometric ratio of a compound angle (using the equations from right to left).

WORKED EXAMPLE 5: COMPOUND ANGLE FORMULAE

QUESTION

Prove that $\sin 75° = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$ without using a calculator.

SOLUTION

Step 1: Consider the given identity

We know the values of the trigonometric functions for the special angles $(30, 45, 60$, etc.$)$ and we can write $75° = 30° + 45°$.

Therefore, we can use the compound angle formula for$\sin(\alpha + \beta)$ to express $\sin 75°$ in terms of known trigonometric function values.

Step 2: Prove the left-hand side of the identity equals the right-hand side

When proving an identity is true, remember to only work with one side of the identity at a time.

$$\begin{aligned}
\text{LHS} &= \sin 75° \\
&= \sin(45° + 30°) \\
\sin(45° + 30°) &= \sin(45°)\cos(30°) + \cos(45°)\sin(30°) \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}(\sqrt{3}+1)}{4} \\
&= \text{RHS}
\end{aligned}$$

Therefore, we have shown that $\sin 75° = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$

WORKED EXAMPLE 6: COMPOUND ANGLE FORMULAE

QUESTION

Determine the value of the following expression without the use of a calculator:

$$\cos 65^\circ \cos 35^\circ + \cos 25^\circ \cos 55^\circ$$

SOLUTION

Step 1: Use co-functions to simplify the expression

- We need to change two of the trigonometric functions from cosine to sine so that we can apply the compound angle formula.
- We also need to make sure that the sum (or difference) of the two angles is equal to a special angle so that we can determine the value of the expression without using a calculator. Notice that $35^\circ + 25^\circ = 60^\circ$.

$$\begin{aligned}&\cos 65^\circ \cos 35^\circ + \cos 25^\circ \cos 55^\circ \\ &= \cos(90^\circ - 25^\circ) \cos 35^\circ + \cos 25^\circ \cos(90^\circ - 35^\circ) \\ &= \sin 25^\circ \cos 35^\circ + \cos 25^\circ \sin 35^\circ\end{aligned}$$

Step 2: Apply the compound angle formula and use special angles to evaluate the expression

$$\begin{aligned}&\sin 25^\circ \cos 35^\circ + \cos 25^\circ \sin 35^\circ \\ &= \sin(25^\circ + 35^\circ) \\ &= \sin 60^\circ\end{aligned}$$

Step 3: Write the final answer

$$\cos 65^\circ \cos 35^\circ + \cos 25^\circ \cos 55^\circ = \frac{\sqrt{3}}{2} \qquad (1)$$

Checking answers: It is always good to check answers. The question stated that we could not use a calculator to find the answer, but we can use a calculator to check that the answer is correct:

$$\begin{aligned}\text{LHS} &= \cos 65^\circ \cos 35^\circ + \cos 25^\circ \cos 55^\circ = 0,866\ldots \\ \text{RHS} &= \frac{\sqrt{3}}{2} = 0,866\ldots \\ \therefore \text{LHS} &= \text{RHS}\end{aligned}$$

3 DOUBLE ANGLE IDENTITIES

3.1 Derivation of $\sin 2\alpha$

We have shown that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. If we let $\alpha = \beta$, then we can write the formula as:

$$\begin{aligned} \sin(2\alpha) &= \sin(\alpha + \alpha) \\ &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\ \therefore \sin 2\alpha &= 2\sin\alpha\cos\alpha \end{aligned}$$

3.2 Derivation of $\cos 2\alpha$

Similarly, we know that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. If we let $\alpha = \beta$, then we have:

$$\begin{aligned} \cos(2\alpha) &= \cos(\alpha + \alpha) \\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\ \therefore \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \end{aligned}$$

And

$$\begin{aligned} \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= \cos^2\alpha - \left(1 - \cos^2\alpha\right) \\ &= \cos^2\alpha - 1 + \cos^2\alpha \\ \therefore \cos 2\alpha &= 2\cos^2\alpha - 1 \end{aligned}$$

Double angle formulae

- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
- $\cos 2\alpha = 1 - 2\sin^2\alpha$
- $\cos 2\alpha = 2\cos^2\alpha - 1$

WORKED EXAMPLE 7: DOUBLE ANGLE IDENTITIES

QUESTION

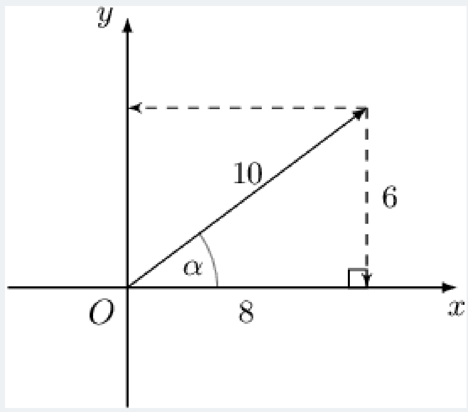
If α is an acute angle and $\sin\alpha = 0,6$, determine the value of $\sin 2\alpha$ without using a calculator.

SOLUTION

Step 1: Draw a sketch

We convert $0,6$ to a fraction so that we can use the ratio to represent the sides of a triangle.

$$\begin{aligned}\sin\alpha &= 0,6 \\ &= \frac{6}{10}\end{aligned}$$



Step 2: Use the double angle formula to determine the value of $\sin 2\alpha$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha \\ &= 2\left(\frac{6}{10}\right)\left(\frac{8}{10}\right) \\ &= \frac{96}{100} \\ &= 0,96\end{aligned}$$

Step 3: Write the final answer

$$\sin 2\alpha = 0,96$$

Check the answer using a calculator:

$$\begin{aligned}\sin\alpha &= 0,6 \\ \therefore \alpha &\approx 36,87^\circ \\ 2\alpha &\approx 73,74^\circ \\ \therefore \sin(73,74^\circ) &\approx 0,96\end{aligned}$$

WORKED EXAMPLE 8: DOUBLE ANGLE IDENTITIES

QUESTION

Prove that $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

For which values of θ is the identity not valid?

SOLUTION

Step 1: Consider the given expressions

The right-hand side (RHS) of the identity cannot be simplified, so we simplify the left-hand side (LHS). We also notice that the trigonometric function on the RHS does not have a 2θ dependence, therefore we will need to use the double angle formulae to simplify $\sin 2\theta$ and $\cos 2\theta$ on the LHS.

Step 2: Prove the left-hand side equals the right-hand side

$$\begin{aligned}
\text{LHS} &= \frac{\sin\theta + 2\sin\theta\cos\theta}{1+\cos\theta+(2\cos^2\theta-1)} \\
&= \frac{\sin\theta\,(1+2\cos\theta)}{\cos\theta\,(1+2\cos\theta)} \qquad \text{(factorise)} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta \\
&= \text{RHS}
\end{aligned}$$

Step 3: Identify restricted values of θ

We know that $\tan\theta$ is undefined for $\theta = 90° + k\cdot 180°, k \in \mathbb{Z}$.

Note that division by zero on the LHS is not allowed, so the identity will also be undefined for:

$$\begin{aligned}
1+\cos\theta+\cos 2\theta &= 0 \\
\cos\theta\,(1+2\cos\theta) &= 0 \\
\therefore \cos\theta = 0 &\text{ or } 1+2\cos\theta = 0
\end{aligned}$$

$$\text{For } \cos\theta = 0, \quad \theta = 90° + k\cdot 180°$$

$$\begin{aligned}
\text{For } 1+2\cos\theta = 0, \quad \cos\theta &= -\frac{1}{2} \\
\therefore \theta &= 120° + k\cdot 360° \\
\text{or } \theta &= 240° + k\cdot 360° \quad \text{for } k \in \mathbb{Z}.
\end{aligned}$$

4 SOLVING EQUATIONS

4.1 The general solution

The periodicity of the trigonometric functions means that there are an infinite number of positive and negative angles that satisfy an equation. If we do not restrict the solution, then we need to determine the general solution to the equation. We know that the sine and cosine functions have a period of $360°$ and the tangent function has a period of $180°$.

Method for finding the solution:

1. Simplify the equation using algebraic methods and trigonometric identities.
2. Determine the reference angle (use a positive value).
3. Use the CAST diagram to determine where the function is positive or negative (depending on the given equation/information).
4. Restricted values: find the angles that lie within a specified interval by adding/subtracting multiples of the appropriate period.
5. General solution: find the angles in the interval $[0°; 360°]$ that satisfy the equation and add multiples of the period to each answer.
6. Check answers using a calculator.

General solutions:

1.

$$\begin{aligned} \text{If } \sin\theta &= x \\ \theta &= \sin^{-1} x + k \cdot 360° \\ \text{or } \theta &= \left(180° - \sin^{-1} x\right) + k \cdot 360° \end{aligned}$$

2.

$$\begin{aligned} \text{If } \cos\theta &= x \\ \theta &= \cos^{-1} x + k \cdot 360° \\ \text{or } \theta &= \left(360° - \cos^{-1} x\right) + k \cdot 360° \end{aligned}$$

3.

$$\begin{aligned} \text{If } \tan\theta &= x \\ \theta &= \tan^{-1} x + k \cdot 180° \qquad \text{for } k \in \mathbb{Z}. \end{aligned}$$

WORKED EXAMPLE 9: FINDING THE GENERAL SOLUTION

QUESTION

Determine the general solution for $\sin\theta = 0,3$ (correct to one decimal place).

SOLUTION

Step 1: Use a calculator to find the reference angle

$$\begin{aligned} \sin\theta &= 0,3 \\ \therefore \text{ref } \angle &= \sin^{-1} 0,3 \\ &= 17,5^\circ \end{aligned}$$

Step 2: Use a CAST diagram to determine in which quadrants $\sin\theta$ is positive

The CAST diagram indicates that $\sin\theta$ is positive in the first and second quadrants.

Using reduction formulae, we know that $\sin(180^\circ - \theta) = \sin\theta$.

In the first quadrant:

$$\begin{aligned} \theta &= 17,5^\circ \\ \therefore \theta &= 17,5^\circ + k \cdot 360^\circ \end{aligned}$$

In the second quadrant:

$$\begin{aligned} \theta &= 180^\circ - 17,5^\circ \\ \therefore \theta &= 162,5^\circ + k \cdot 360^\circ \end{aligned}$$

where $k \in \mathbb{Z}$.

Step 3: Check that the solution satisfies the original equation

We can select random values of k to check that the answers satisfy the original equation.

Let $k = 4$

$$\begin{aligned} \theta &= 17,5^\circ + 4(360)^\circ \\ \therefore \theta &= 1\,457,5^\circ \\ \text{And } \sin 1\,457,5^\circ &= 0,3007\ldots \end{aligned}$$

This solution is correct.

Similarly, if we let $k = -2$:

$$\begin{aligned} \theta &= 162,5^\circ - 2(360)^\circ \\ \therefore \theta &= -557,5^\circ \\ \text{And } \sin(-557,5^\circ) &= 0,3007\ldots \end{aligned}$$

This solution is also correct.

Step 4: Write the final answer

$\theta = 17,5^\circ + k \cdot 360^\circ$ or $\theta = 162,5^\circ + k \cdot 360^\circ$ for $k \in \mathbb{Z}$.

WORKED EXAMPLE 10: TRIGONOMETRIC EQUATIONS

QUESTION

Solve the following equation for y, without using a calculator:

$\frac{1-\sin y-\cos 2y}{\sin 2y-\cos y} = -1$

SOLUTION

Step 1: Simplify the equation

We first simplify the left-hand side of the equation using the double angle formulae. To solve this equation, we need to manipulate the given equation to be of the form:

$$\text{single trigonometric ratio} = \text{constant}$$

$$\begin{aligned}
\frac{1-\sin y-\left(1-2\sin^2 y\right)}{2\sin y\cos y-\cos y} &= -1 \\
\frac{2\sin^2 y-\sin y}{\cos y\,(2\sin y-1)} &= -1 \\
\frac{\sin y\,(2\sin y-1)}{\cos y\,(2\sin y-1)} &= -1 \\
\frac{\sin y}{\cos y} &= -1 \\
\therefore \tan y &= -1
\end{aligned}$$

Step 2: Use a calculator to find the reference angle

$$\begin{aligned}
\tan y &= -1 \\
\therefore \text{ref } \angle &= \tan^{-1}(1) \\
&= 45^\circ
\end{aligned}$$

Step 3: Use CAST diagram to determine in which quadrants $\tan y$ **is negative**

The CAST diagram indicates that $\tan y$ is negative in the second and fourth quadrants.

$$\begin{aligned}
y &= (180^\circ - 45^\circ) + k \cdot 180^\circ \\
\therefore y &= 135^\circ + k \cdot 180^\circ, \quad k \in \mathbb{Z}
\end{aligned}$$

Notice that for $k = 1$, $y = 135^\circ + 180^\circ = 315^\circ$, which is the angle in the fourth quadrant.

Step 4: Check that the solution satisfies the original equation

Step 5: Write the final answer

$y = 135^\circ + k \cdot 180^\circ$ where $k \in \mathbb{Z}$.

WORKED EXAMPLE 11: TRIGONOMETRIC EQUATIONS

QUESTION

Prove $8\cos^4 x - 8\cos^2 x + 1 = \cos 4x$ and hence solve $8\cos^4 x - 8\cos^2 x + 1 = 0,8$ (correct to one decimal place).

SOLUTION

Step 1: Prove the identity

Expand the right-hand side of the identity and show that it is equal to the left-hand side:

$$\begin{aligned} \text{RHS} &= \cos 4x \\ &= 2\cos^2 2x - 1 \\ &= 2\left(2\cos^2 x - 1\right)^2 - 1 \\ &= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1 \\ &= 8\cos^4 x - 8\cos^2 x + 2 - 1 \\ &= 8\cos^4 x - 8\cos^2 x + 1 \\ &= \text{LHS} \end{aligned}$$

Step 2: Solve the equation

$$\begin{aligned} 8\cos^4 x - 8\cos^2 x + 1 &= 0,8 \\ \therefore \cos 4x &= 0,8 \end{aligned}$$

Step 3: Use a calculator to find the reference angle

$$\begin{aligned} \cos 4x &= 0,8 \\ \therefore \text{ref } \angle &= \cos^{-1}(0,8) \\ &= 36,9^\circ \end{aligned}$$

Step 4: Use CAST diagram to determine in which quadrants $\cos 4x$ is positive

The CAST diagram indicates that $\cos 4x$ is positive in the first and fourth quadrants.

In the first quadrant:

$$\begin{aligned} 4x &= 36,9^\circ + k \cdot 360^\circ \\ \therefore x &= 9,2^\circ + k \cdot 90^\circ \end{aligned}$$

Important: remember to also divide $k \cdot 360^\circ$ by 4.

In the fourth quadrant:

$$\begin{aligned} 4x &= (360° - 36,9°) + k \cdot 360° \\ &= 323,1° + k \cdot 360° \\ \therefore x &= 80,8° + k \cdot 90° \end{aligned}$$

where $k \in \mathbb{Z}$.

Step 5: Check that the solution satisfies the original equation

Step 6: Write the final answer

$$\begin{aligned} x &= 9,2° + k \cdot 90° \\ \text{or } x &= 80,8° + k \cdot 90°, \quad k \in \mathbb{Z} \end{aligned}$$

WORKED EXAMPLE 12: TRIGONOMETRIC EQUATIONS

QUESTION

Find the general solution for $\sin\theta \cos^2\theta = \sin^3\theta$.

SOLUTION

Step 1: Simplify the given equation

Do not divide both sides of the equation by $\sin\theta$.

- part of the solution would be lost;
- we would need to restrict the values of θ to those where $\sin\theta \neq 0$ (division by zero is not permitted).

$$\begin{aligned} \sin\theta\cos^2\theta &= \sin^3\theta \\ \sin\theta\cos^2\theta - \sin^3\theta &= 0 \\ \sin\theta\left(\cos^2\theta - \sin^2\theta\right) &= 0 \\ \sin\theta\left(\cos\theta - \sin\theta\right)\left(\cos\theta + \sin\theta\right) &= 0 \end{aligned}$$

Step 2: Apply the zero product law and solve for θ

$$\sin\theta\left(\cos\theta - \sin\theta\right)\left(\cos\theta + \sin\theta\right) = 0$$

$$\begin{aligned} \sin\theta &= 0 \\ \therefore \theta &= 0° + k \cdot 360° \\ \text{or } \theta &= 180° + k \cdot 360°, \quad k \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \cos\theta - \sin\theta &= 0 \\ \cos\theta &= \sin\theta \\ \cos\theta &= \cos(90^\circ - \theta) \\ \therefore \theta &= (90^\circ - \theta) + k \cdot 360^\circ \\ 2\theta &= 90^\circ + k \cdot 360^\circ \\ \therefore \theta &= 45^\circ + k \cdot 180^\circ, \quad k \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \cos\theta + \sin\theta &= 0 \\ \cos\theta &= -\sin\theta \\ \sin(90^\circ + \theta) &= \sin(-\theta) \\ \therefore 90^\circ + \theta &= -\theta + k \cdot 360^\circ \\ 2\theta &= -90^\circ + k \cdot 360^\circ \\ \therefore \theta &= -45^\circ + k \cdot 180^\circ, \quad k \in \mathbb{Z} \end{aligned}$$

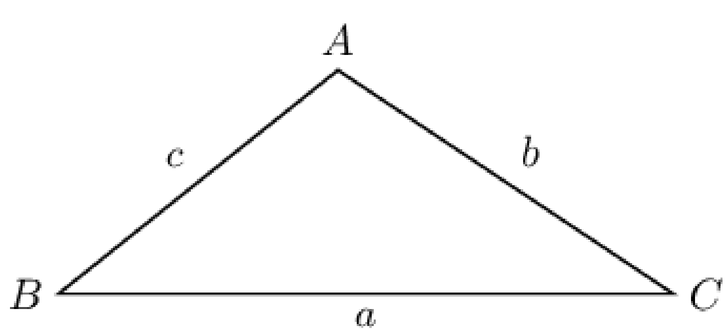
Step 3: Write the final answer

$$\begin{aligned} \theta &= 0^\circ + k \cdot 360^\circ \\ \text{or } \theta &= 180^\circ + k \cdot 360^\circ \\ \text{or } \theta &= \pm 45^\circ + k \cdot 180^\circ, \quad k \in \mathbb{Z} \end{aligned}$$

5 APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

5.1 Applications of trigonometric functions

Area, sine and cosine rule



Area rule	Sine rule	Cosine rule
area $\triangle ABC = \frac{1}{2}bc \sin \hat{A}$	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin \hat{C}}{c}$	$a^2 = b^2 + c^2 - 2bc \cos A$
area $\triangle ABC = \frac{1}{2}ac \sin \hat{B}$	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin \hat{C}}$	$b^2 = a^2 + c^2 - 2ac \cos B$
area $\triangle ABC = \frac{1}{2}ab \sin \hat{C}$		$c^2 = a^2 + b^2 - 2ab \cos \hat{C}$

How to determine which rule to use:

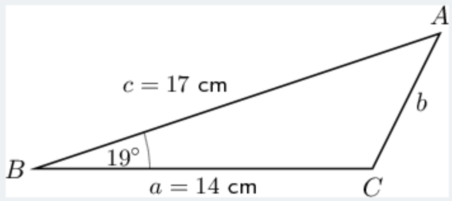
1. Area rule:
   - no perpendicular height is given
2. Sine rule:
   - no right angle is given
   - two sides and an angle are given (**not** the included angle)
   - two angles and a side are given
3. Cosine rule:
   - no right angle is given
   - two sides and the included angle are given
   - three sides are given

5.2 Problems in two dimensions

WORKED EXAMPLE 13: AREA

QUESTION

Given $\triangle ABC$ with $a = 14$ cm, $c = 17$ cm and $\hat{B} = 19°$.



Calculate the following:

1. b
2. $\hat{C}$
3. area $\triangle ABC$

SOLUTION

Step 1:Use the cosine rule to determine the length of b

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\hat{B} \\ &= (14)^2 + (17)^2 - 2(14)(17)\cos 19° \\ &= 34,93\ldots \\ \therefore b &= 5,9 \text{ cm} \end{aligned}$$

Step 2: Use the sine rule to determine $\hat{C}$

$$\begin{aligned} \frac{\sin\hat{C}}{c} &= \frac{\sin\hat{B}}{b} \\ \frac{\sin\hat{C}}{17} &= \frac{\sin 19°}{5,9} \\ \sin\hat{C} &= \frac{17\times\sin 19°}{5,9} \\ \therefore \sin\hat{C} &= 0,938\ldots \\ \text{First quadrant: } \hat{C} &= 69,7° \\ \text{Second quadrant: } \hat{C} &= 180° - 69,7° \\ &= 110,3° \end{aligned}$$

From the diagram, we see that $\hat{C} > 90°$, therefore $\hat{C} = 110,3°$.

Step 3: Calculate the area $\triangle ABC$

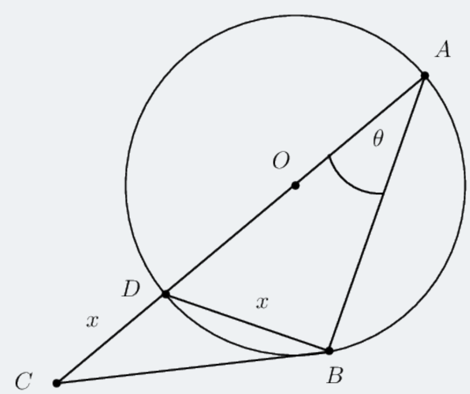
$$\begin{aligned}\text{Area } \triangle ABC &= \frac{1}{2}ac\sin\hat{B}\\ &= \frac{1}{2}(14)(17)\sin 19°\\ &= 38,7\text{ cm}^2\end{aligned}$$

WORKED EXAMPLE 14: PROBLEM IN TWO DIMENSIONS

QUESTION

In the figure below, $CD = BD = x$ and $B\hat{A}D = \theta$.

Show that $BC^2 = 2x^2(1 + \sin\theta)$.



SOLUTION

Step 1: Consider the given information

Use the given information to determine as many of the unknown angles as possible.

$$\begin{aligned}CD &= BD = x && \text{(given)}\\ B\hat{A}D &= \theta && \text{(given)}\\ D\hat{B}A &= 90° && (\angle \text{ in semi circle})\\ B\hat{D}A &= 180° - 90° - \theta && (\angle\text{s sum of } \triangle ABD)\\ &= 90° - \theta\\ B\hat{D}C &= 90° + \theta && (\angle\text{s on a str. line})\end{aligned}$$

Step 2: Determine the expression for BC

To derive the required expression, we need to write BC in terms of x and θ.

In $\triangle CDB$ we can use the cosine rule to determine BC:

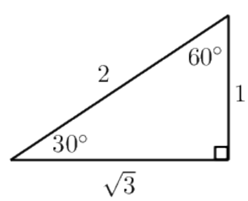
$$\begin{aligned} BC^2 &= CD^2 + BD^2 - 2 \cdot CD \cdot BD \cdot \cos\left(B\hat{D}C\right) \\ &= x^2 + x^2 - 2x^2 \cos\left(90^\circ + \theta\right) \\ &= 2x^2 - 2x^2\left(-\sin\theta\right) \\ &= 2x^2 + 2x^2 \sin\theta \\ &= 2x^2\left(1 + \sin\theta\right) \end{aligned}$$

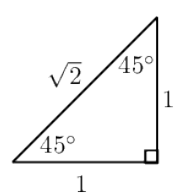
6 SUMMARY

6.1 Summary

Pythagorean Identities	Ratio Identities
$\cos^2\theta + \sin^2\theta = 1$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\cos^2\theta = 1 - \sin^2\theta$	$\frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$
$\sin^2\theta = 1 - \cos^2\theta$	

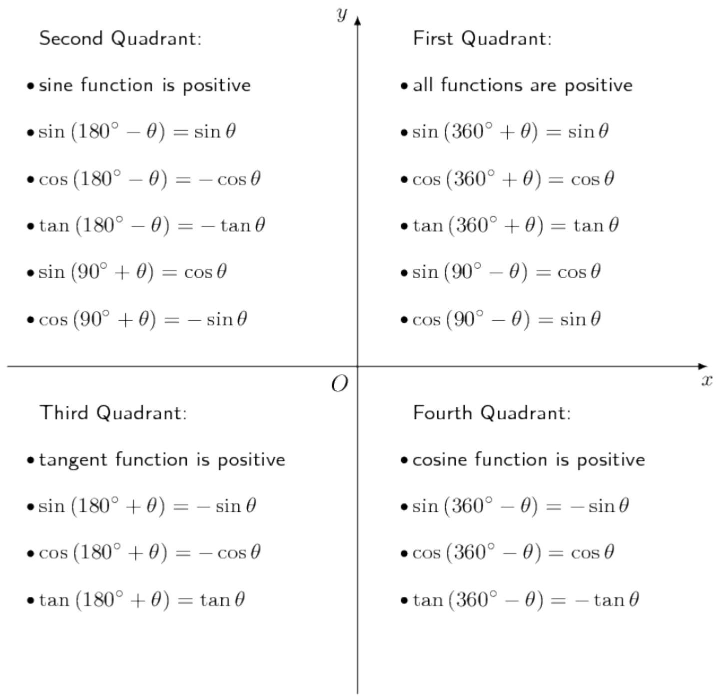
Special angle triangles



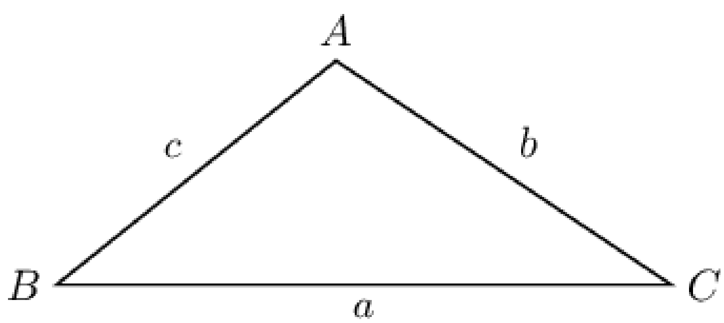


θ	0°	30°	45°	60°	90°
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

CAST diagram and reduction formulae



Negative angles	Periodicity Identities	Cofunction Identities
$\sin(-\theta) = -\sin\theta$	$\sin(\theta \pm 360^\circ) = \sin\theta$	$\sin(90^\circ - \theta) = \cos\theta$
$\cos(-\theta) = \cos\theta$	$\cos(\theta \pm 360^\circ) = \cos\theta$	$\cos(90^\circ - \theta) = \sin\theta$
$\tan(-\theta) = -\tan\theta$	$\tan(\theta \pm 180^\circ) = \tan\theta$	$\sin(90^\circ + \theta) = \cos\theta$
		$\cos(90^\circ + \theta) = -\sin\theta$



Area Rule	Sine Rule	Cosine Rule
Area $= \frac{1}{2}bc \sin \hat{A}$	$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$	$a^2 = b^2 + c^2 - 2bc \cos \hat{A}$
Area $= \frac{1}{2}ab \sin \hat{C}$	$a \sin \hat{B} = b \sin \hat{A}$	$b^2 = a^2 + c^2 - 2ac \cos \hat{B}$
Area $= \frac{1}{2}ac \sin \hat{B}$	$b \sin C = c \sin \hat{B}$	$c^2 = a^2 + b^2 - 2ab \cos \hat{C}$
	$a \sin C = c \sin \hat{A}$	

Compound Angle Identities	Double Angle Identities
$\sin(\theta + \beta) = \sin\theta \cos\beta + \cos\theta \sin\beta$	$\sin(2\theta) = 2\sin\theta\cos\theta$
$\sin(\theta - \beta) = \sin\theta \cos\beta - \cos\theta \sin\beta$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\cos(\theta + \beta) = \cos\theta \cos\beta - \sin\theta \sin\beta$	$\cos(2\theta) = 1 - 2\sin^2\theta$
$\cos(\theta - \beta) = \cos\theta \cos\beta + \sin\theta \sin\beta$	$\cos(2\theta) = 2\cos^2\theta - 1$
	$\tan(2\theta) = \frac{\sin 2\theta}{\cos 2\theta}$

7 EXERCISES

7.1 Exercise 1

1. Given: $\sin 31° = A$

   Write each of the following expressions in terms of A:

   1.1 $\sin 149°$

   1.2 $\cos(-59°)$

   1.3 $\cos 329°$

   1.4 $\tan 211° \cos 211°$

   1.5 $\tan 31°$

2. Simplify the following to a single trigonometric ratio:

   2.1 $P = \sin(360° + \theta)\cos(180° + \theta)\tan(360° + \theta)$

   2.2 $Q = \frac{\cos(\theta - 360°)\sin(90° + \theta)\sin(-\theta)}{\sin(\theta + 180°)}$

   2.3 Hence, determine: $P + Q$

   2.4 $\frac{Q}{P}$

3. If $p = \sin\beta$, express the following in terms of p:

$$\frac{\cos(\beta + 360°)\tan(\beta - 360°)\cos(\beta + 90°)}{\sin^2(\beta + 180°)\cos(\beta - 90°)}$$

4. Evaluate the following without the use of a calculator:

   4.1 $\frac{\cos(-120°)}{\tan 150°} + \cos 390°$

   4.2 $(1 - \sin 45°)(1 - \sin 225°)$

5. Reduce the following to one trigonometric ratio:

   5.1 $\tan^2\beta - \frac{1}{\cos^2\beta}$

   5.2 $\sin^2(90° + \theta)\tan^2\theta + \tan^2\theta\cos^2(90° - \theta)$

   5.3 $\sin\alpha\cos\alpha\tan\alpha - 1$

   5.4 $\tan^2\theta + \frac{\cos^2\theta - 1}{cos^2\theta}$

6. Answer the following:

   6.1 Use reduction formula and special angles to simplify

$$\frac{\sin(180° + \theta)\tan(720° + \theta)\cos(-\theta)}{\cos(90° + \theta)}$$

6.2 Without using a calculator determine the value of $\sin 570°$

7. Troy's mathematics teacher asks the class to answer the following question.

   Prove that:

   $\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$

   Troy's answer:

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$$
$$(\cos\theta)(\cos\theta) = (1+\sin\theta)(1-\sin\theta)$$
$$\cos^2\theta = 1-\sin 2\theta$$
$$\cos^2\theta = \cos^2\theta$$
$$\therefore LHS = RHS$$

   Comment on Troy's answer and show the correct method for proving this identity.

8. Prove the following identities:

   Determine where the identities are undefined.

   8.1 $\sin^2\alpha + (\cos\alpha - \tan\alpha)(\cos\alpha + \tan\alpha) = 1 - \tan^2\alpha$

   8.2 $\frac{1}{\cos\theta} - \frac{\cos\theta\tan^2\theta}{1} = \cos\theta$

   8.3 $\frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta} = \sin\theta + \cos\theta - \frac{1}{\sin\theta+\cos\theta}$

9. Determine whether the following statements are true or false. If the statement is false, choose a suitable value between $0°$ and $90°$ to confirm your answer.

   9.1 $\cos(180° - \theta) = -1 - \cos\theta$

   9.2 $\sin(\alpha+\beta) = \sin\alpha + \sin\beta$

   9.3 $\sin\alpha = \sin\frac{\alpha}{2}\sin\frac{\alpha}{2}$

   9.4 $\frac{1}{3}\sin 3\alpha = \sin\alpha$

   9.5 $\cos\beta = \sqrt{1-\sin^2\beta}$

   9.6 $\sin\theta = \tan\theta\cos\theta$

10. Given the graphs of $f(\theta) = p\sin k\theta$ and $g(\theta) = q\tan\theta$, determine the values of p, k and q.

7.2 Exercise 2

1. Given:

$$13\sin\alpha + 5 = 0\ (0° < \alpha < 270°)$$
$$13\cos\beta - 12 = 0\ (90° < \beta < 360°)$$

Draw a sketch and determine the following without the use of a calculator:

1.1 $\tan\alpha - \tan\beta$

1.2 $\sin(\beta - \alpha)$

1.3 $\cos(\alpha + \beta)$

2. Calculate the following without the use of an calculator. (Leave answers in surd form):

2.1 $\sin 105°$

2.2 $\cos 15°$

2.3 $\sin 15°$

2.4 $\tan 15°$

2.5 $\cos 20° \cos 40° - \sin 20° \sin 40°$

2.6 $\sin 10° \cos 80° + \cos 10° \sin 80°$

2.7 $\cos(45° - x)\cos x - \sin(45° - x)\sin x$

2.8 $\cos^2 15° - \sin^2 15°$

3. Answer the following:

3.1 Prove: $\sin(60° - x) + \sin(60° + x) = \sqrt{3}\cos x$

3.2 Hence, evaluate $\sin 15° + \sin 105°$ without using a calculator.

3.3 Use a calculator to check your answer.

4. Simplify the following without using a calculator:

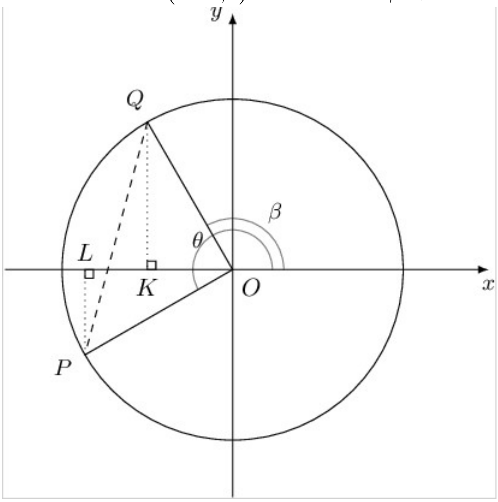
$$\frac{\sin p \cos(45°-p)+\cos p \sin(45°-p)}{\cos p \cos(60°-p)-\sin p \sin(60°-p)}$$

5. Answer the following:

5.1 Prove: $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$

5.2 Hence, calculate the value of $\cos 75° \sin 15°$ without using a calculator.

6. In the diagram below, Points P and Q lie on the circle with radius of 2 units and centre at the origin. Prove that $\cos(\theta - \beta) = \cos\theta\cos\beta + \sin\theta\sin\beta$



7. Prove: $\cos(90° + x) = -\sin x$

7.3 Exercise 3

1. Given $5\cos\theta = -3$ and $\theta < 180°$, determine the following without the use of a calculator:

   1.1 $\cos 2\theta$

   1.2 $\sin(180° - 2\theta)$

   1.3 $\tan 2\theta$

   1.4 $\sin 2\theta$

   1.5 $\cos(90° + 2\theta)$

2. Given $\cos 40° = t$, determine (without a calculator):

   2.1 $\cos 140°$

   2.2 $\sin 40°$

   2.3 $\sin 50°$

   2.4 $\cos 80°$

   2.5 $\cos 860°$

   2.6 $\cos(-1160°)$

   2.7 $\sin 80°$

3. Answer the following:

   3.1 Prove the identity: $\frac{1}{\sin 2A} - \frac{1}{\tan 2A} = \tan A$

   3.2 Hence, solve the equation: $\frac{1}{\sin 2A} - \frac{1}{\tan 2A} = 0{,}75$ for $0° < A < 360°$

4. Without using a calculator, find the value of the following:

   4.1 $\sin 22{,}5°$

   4.2 $\cos 67{,}5°$

   4.3 $\sin 67{,}5°$

5. Answer the following:

   5.1 Prove the identity: $\tan 2x + \frac{1}{\cos 2x} = \frac{\sin x + \cos x}{\cos x - \sin x}$

   5.2 Explain why the identity is undefined for $x = 45°$

7.4 Exercise 4

1. Prove the following identity:

   1.1 $\tan y = \frac{\sin 2y}{\cos 2y + 1}$

   1.2 $\frac{\cos 2x}{\sin 2x + 1} = \frac{\cos x - \sin x}{\cos x + \sin x}$

   1.3 $\tan A + \sin 2A = \tan A(1 + 2\cos^2 A)$

   1.4 $\frac{1 - 2\cos^2 x}{\frac{1}{\tan 2x}} = -2\sin x \cos x$

   1.5 $\cos 4\theta = 4\cos^4\theta - 2\cos^2\theta - 1$

   1.6 $\frac{\frac{1}{\tan A} + 1}{\tan A + 1} = \frac{1}{\tan A}$

   1.7 $\frac{1 - \cos 4\beta}{\sin 2\beta} = \tan \beta$

   1.8 $1 + 2\sin 2x + 2\sin^2 x = (\cos x + \sin x)(\cos x + 3\sin x)$

   1.9 $\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} = \frac{\sin 2x}{\cos^4 x \tan x}$

2. Determine where the identities are undefined.

   2.1 $\tan y = \frac{\sin 2y}{\cos 4y + 1}$

   2.2 $\frac{\cos 4x}{\sin 2x + 1} = \frac{\cos x - \sin x}{\cos x + \sin x}$

   2.3 $\tan A + \sin 2A = \tan A \left(1 + 2\cos^2 A\right)$

   2.4 $\frac{1 - 2\cos^2 x}{\frac{1}{\tan 2x}} = -2\sin x \cos x$

   2.5 $\cos 4\theta = 4\cos^4\theta - 2\cos^2\theta - 1$

   2.6 $\frac{\frac{\cos A}{\tan A} - \sin A}{1 + \frac{1}{\tan A}} = \cos A - \sin A$

   2.7 $\frac{\frac{1}{\tan A} + 1}{\tan A + 1} = \frac{1}{\tan A}$

2.8 $\frac{1-\cos 4\beta}{\sin 2\beta} = \tan \beta$

2.9 $1 + 2\sin 2x + 2\sin^2 x = (\cos x + \sin x)(\cos x + 3\sin x)$

2.10 $\frac{1}{1+\sin x} + \frac{1}{1-\sin x} = \frac{\sin 2x}{\cos^4 x \tan x}$

7.5 Exercise 5

1. Find the general solution for each of the following equations (correct to two decimal places):

    1.1 $\sin 2x = \tan 28°$

    1.2 $\sin 2\alpha = \cos 2\alpha$

    1.3 $\tan A = \frac{1}{\tan A}$

    1.4 $3\cos(x - 30°) = 2$

    1.5 $\sin x \tan x = 1$

    1.6 Solve for x: $\sqrt{3}\sin x + \cos x = 2$

2. Find the general solution for each of the following equations (correct to two decimal places):

    2.1 $\cos y = \sin 2y$

    2.2 $\sin 3p = \sin 2p$

    2.3 $\cos(x + 20°) = \sin(2x - 10°)$

    2.4 $\tan 2A = -\tan A$

    2.5 $\cos(\beta - 25°) = \cos 2\beta$

    2.6 $\sin(x + 30°) = -\cos 2x$

3. Find the general solution for each of the following equations (correct to two decimal places):

    3.1 $\sin t \cdot \sin 2t + \cos 2t = 1$

    3.2 $\sin 60° \cos x + \cos 60° \sin x = 1$

    3.3 $\cos A \cdot \cos 30° - \sin A \cdot \sin 30° = 0{,}62$

4. Without using a calculator, solve $\cos(A - 25°) + \cos(A + 25°) = \cos 25°$ in $[-360°; 360°]$

5. Solve the following:

    5.1 Find the general solution for $\sin x \cos 3x + \cos x \sin 3x = \tan 140°$

    5.2 Use a graph to illustrate the solution for the interval $[0°; 90°]$

6. Given that $\sin(x + 60°) = -\frac{1}{2}\sin x$, prove that $\tan x = -\frac{\sqrt{3}}{2}$

7. Given that $\sin x \cos x = \sqrt{3} \sin^2 x$

   7.1 Solve the equation for $x \, \epsilon \, [0°; 360°]$ without using a calculator.

   7.2 Draw a graph and indicate the solution on the diagram.

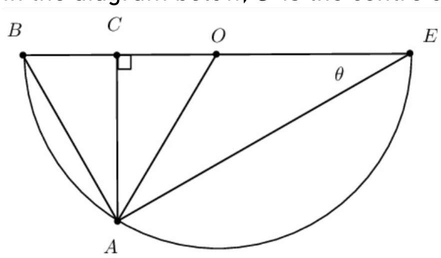
8. Given: $1 + \tan^2 2A = 5 \tan 2A - 5$

   8.1 Determine the general solution.

   8.2 How many solutions does the given equation have in the interval $[-90°; 360°]$?

7.6 Exercise 6

1. In the diagram below, O is the centre of the semi-circle BAE.



   1.1 Find $A\hat{O}C$ in terms of θ

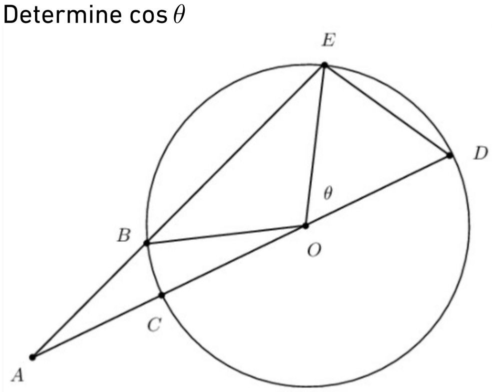
   1.2 In $\triangle ABE$, determine an expression for $\cos \theta$

   1.3 In $\triangle ACE$, determine an expression for $\sin \theta$

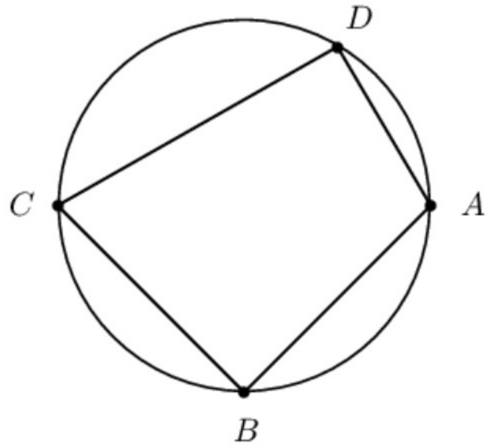
   1.4 In $\triangle ACO$, determine an expression for $\sin 2\theta$

   1.5 Use the answers from the previous questions to expand $\sin 2\theta$

2. DC is a diameter of the circle with the centre O and radius r. $CA = r$, $AE = 2DE$ and $D\hat{O}E = \theta$. Determine $\cos \theta$



3. The figure below shows a cyclic quadrilateral with $\frac{BC}{CD} = \frac{AD}{AB}$



3.1 Determine the area of the cyclic quadrilateral.

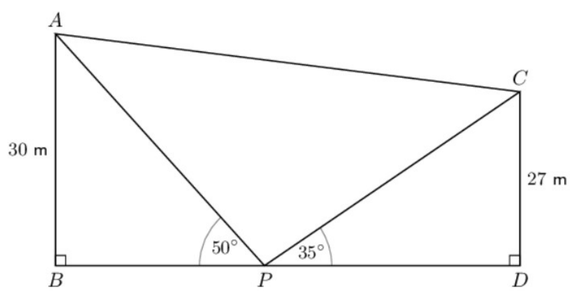
3.2 Write down two expressions for CA^2: one in terms of cos $\hat{D}$ and one in terms of cos $\hat{B}$.

3.3 Determine $2CA^2$

3.4 Suppose that $BC = 10$ units, $CD = 15$ units, $AD = 4$ units and $AB = 6$ units. Calculate CA^2 (correct to one decimal place).

3.5 Find the angle $\hat{B}$. Hence, calculate the area of $ABCD$ (correct to one decimal place).

4. Two vertical towers AB and CD are 30m and 27m high, respectively. Point P lies between the two towers. The angle of elevation from P to A is $50°$ and from P to C is $35°$. A cable is needed to connect A and C.

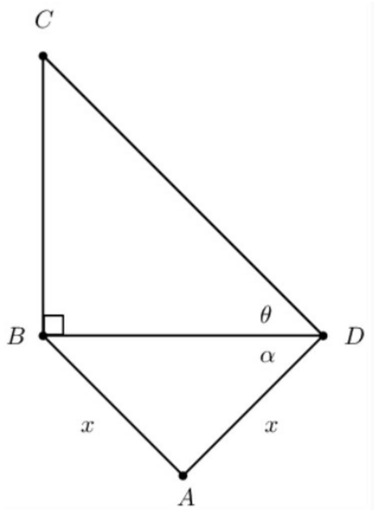


4.1 Determine the minimum length of cable needed to connect A and C (to the nearest metre).

4.2 How far apart are the bases of the two towers (to the nearest metre)?

7.7 Exercise 7

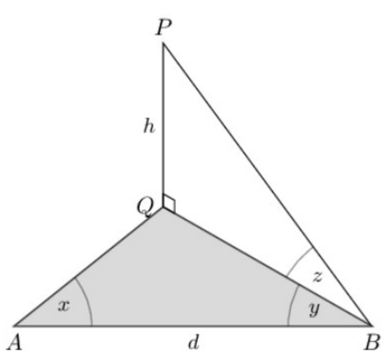
1. The line BC represents a tall tower, with B at its base. The angle of elevation from D to C is θ. A man stands at A such that $BA = AD = x$ and $A\hat{D}B = \alpha$



   1.1 Find the height of the tower BC in terms of x, $\tan\theta$ and $\cos\alpha$

   1.2 Find BC if we are given that $x = 140$ m, $\alpha = 21°$ and $\theta = 9°$

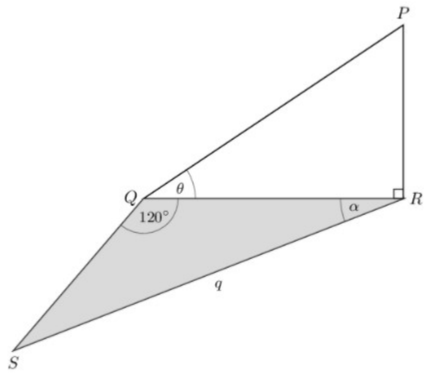
2. P is the top of a mast and its base, Q, is in the same horizontal plane as the points A and B. The angle of elevation measured from B to P is z. $AB = d$, $Q\hat{A}B = x$ and $Q\hat{B}A = y$.



   2.1 Use the given information to derive a general formula for h, the height of the mast.

   2.2 If $d = 50$m, $x = 46°$, $y = 15°$ and $z = 20°$, calculate h (to the nearest metre).

3. PR is the height of a block of flats with R at the base and P at the top of the building. S is a point in the same horizontal plane as points Q and R. $SR = q$ units, $S\hat{Q}R = 120°$, $S\hat{R}Q = \alpha$ and $R\hat{Q}P = \theta$

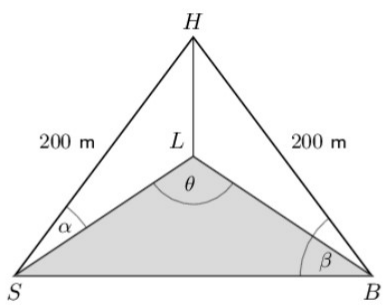


3.1 Prove that the height of the block of flats, PR, can be expressed as: $PR = q \tan\theta \left(\cos\alpha - \frac{\sqrt{3}\sin\alpha}{3}\right)$

3.2 If $SR = 35$m, $S\hat{R}Q = 16°$ and $R\hat{Q}P = 30°$ calculate PR (correct to one decimal place).

3.3 Assuming each level is $2{,}5$ m high, estimate the number of levels in the block of flats.

4. Two ships at sea can see a lighthouse on the shore. The distance from the top of the lighthouse (H) to ship S and to ship B is 200 m. The angle of elevation from S to H is α, $H\hat{B}S = \beta$ and $S\hat{L}B = \theta$

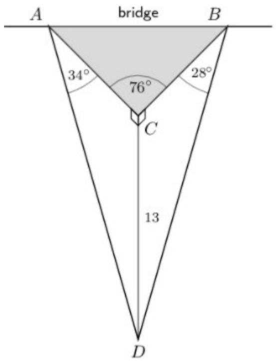


4.1 Determine the distance between the two ships in terms of β.

4.2 Determine the area of the sea included in ΔLSB.

4.3 Calculate the triangular area of the sea if the angle of inclination from the ship to the top of the lighthouse is $10°$ and the angle between the direct lines from the base of the lighthouse to each ship is $85°$.

5. A triangular look-out platform (ΔABC) is attached to a bridge that extends over a deep gorge. The vertical depth of the gorge, the distance from the edge of the look-out C to the bottom of the gorge D, is 13 m. The angle of depression from A to D is $34°$ and from B to D is $28°$. The angle at the edge of the platform, $\hat{C}$ is $76°$



5.1 Calculate the area of the look-out platform (to the nearest m^2)

5.2 If the platform is constructed so that the two angles of depression, $C\hat{A}D$ and $C\hat{B}D$ are both equal to $45°$ and the vertical depth of the gorge $CD = d$, $AB = x$ and $A\hat{C}B = \theta$, determine cos θ

5.3 If $AB = 25$m and $CD = 13$m, calculate $A\hat{C}B$

8 ANSWERS FOR EXERCISES

8.1 Exercise 1

1.1 $\sin 149° = A$

1.2 $\cos(-59°) = A$

1.3 $\cos 329° = \sqrt{1 - A^2}$

1.4 $\tan 211° \cos 211° = -A$

1.5 $\tan 31° = \frac{A}{\sqrt{1-A^2}}$

2.1 $P = \sin^2 \theta$

2.2 $Q = \cos^2 \theta$

2.3 $P + Q = 1$

2.4 $\frac{Q}{P} = \frac{1}{\tan^2 \theta}$

$-\frac{1}{p}$

4.1 $\sqrt{3}$

4.2 $\frac{1}{2}$

5.1 -1

5.2 $\tan^2 \theta$

5.3 $-\cos^2 \alpha$

5.4 0

6.1 $\sin \theta$

6.2 $-\frac{1}{2}$

7. Incorrect method. By working with both sides of the identity at the same time, he accepted that it was true. The correct method for proving an identity is to work with only one side at a time and to show that one side equals the other.

8.1 $\alpha = 90°; 270°$

8.2 $\theta = 90°; 270°$

8.3 $\theta = 0°; 90°; 180°; 270°; 360°$

9.1 False

9.2 False

9.3 False

9.4 False

9.5 True

9.6 True

10. $f(\theta) = \frac{3}{2}\sin 2\theta$ and $g(\theta) = -\frac{3}{2}\tan\theta$

8.2 Exercise 2

1.1 $\frac{5}{6}$

1.2 $\frac{120}{169}$

1.3 -1

2.1 $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$

2.2 $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$

2.3 $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$

2.4 $2-\sqrt{3}$

2.5 $\frac{1}{2}$

2.6 1

2.7 $\frac{\sqrt{2}}{2}$

2.8 $\frac{\sqrt{3}}{2}$

3.1 LHS

$= \sin(60° - x) + \sin(60° + x)$

$= \sin 60° \cos x - \cos 60° \sin x + \sin 60° \cos x + \cos 60° \sin x$

$= 2\sin 60° \cos x$

$= 2(\frac{\sqrt{3}}{2})\cos x$

$= \sqrt{3}\cos x$

$= RHS$

3.2 $\frac{\sqrt{6}}{2}$

3.3 $LHS = RHS = 1{,}2247$

4. $\sqrt{2}$

5.1 LHS
$= \sin(A + B) - \sin(A - B)$
$= \sin A \cos B + \cos A \sin B - [\sin A \cos B - \cos A \sin B]$
$= \sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B$
$= 2 \cos A \sin B$
$= RHS$

5.2 $\frac{2-\sqrt{3}}{4}$

6. We can express the coordinates of P and Q in terms of the angles θ and β:
For Q, $\sin \beta = \frac{y}{2}$
Hence, $y = 2 \sin \beta$
and $x = 2 \cos \beta$
Therefore, $Q(2 \cos \beta; 2 \sin \beta)$

Similarly, $P(2 \cos \theta; 2 \sin \theta)$
We use the distance formula to determine PQ^2:
In ΔPOQ:
$P\hat{O}Q = \theta - \beta$ $d^2 = (x_P - x_Q)^2 + (y_P - y_Q)^2$
$PQ^2 = (2 \cos \theta - 2 \cos \beta)^2 + (2 \sin \theta - 2 \sin \beta)^2$
$= 4 \cos^2 \theta - 8 \cos \theta \cos \beta + 4 \cos^2 \beta + 4 \sin^2 \theta - 8 \sin \theta \sin \beta + 4 \sin^2 \beta$
$= 4(\cos^2 \theta + \sin^2 \theta) + 4(\cos^2 \beta + \sin^2 \beta) - 8 \cos \theta \cos \beta - 8 \sin \theta \sin \beta$
$= 8 - 8(\cos \theta \cos \beta + \sin \theta \sin \beta)$
Now we determine PQ^2 using the cosine rule for ΔPOQ
$PQ^2 = 2^2 + 2^2 - 2(2)(2) \cos(\theta - \beta)$
$= 8 - 8 \cos(\theta - \beta)$
Equating the two expressions for PQ^2, we have
$8 - 8 \cos(\theta - \beta) = 8 - 8(\cos \theta \cos \beta + \sin \theta \sin \beta)$
Hence, $\cos(\theta - \beta) = 0 \cos \theta \cos \beta + \sin \theta \sin \beta$
Note: earlier in this chapter we derived the compound angle identities using a unit circle (radius = 1 unit) because it simplified the calculations. From the exercise above, we see that the compound angle identities can in fact be derived using a radius of any length.

7. LHS
$= \cos(90° + x)$
$= \cos 90° \cos x - \sin 90° \sin x$
$= (0) \cos x - (1) \sin x$
$= -\sin x$
$= RHS$

8.3 Exercise 3

1.1 $-\frac{7}{25}$

1.2 $-\frac{24}{25}$

1.3 $\frac{24}{7}$

1.4 $-\frac{24}{25}$

1.5 $\frac{24}{25}$

2.1 $-t$

2.2 $\sqrt{1-t^2}$

2.3 t

2.4 $2t^2-1$

2.5 $-t$

2.6 $2t^2-1$

2.7 $2t\sqrt{1-t^2}$

3.1 LHS

$= \frac{1}{\sin 2A} - \frac{1}{\tan 2A}$

$= \frac{1}{\sin 2A} - \frac{\cos 2A}{\sin 2A}$

$= \frac{1-(1-2\sin^2 A)}{\sin 2A}$

$= \frac{2\sin^2 A}{2\sin A\cos A}$

$= \frac{\sin A}{\cos A}$

$= \tan A$

$= RHS$

Restrictions:

$\sin 2A \neq 0$

$\therefore 2A \neq 0° + k \cdot 180°$

$\therefore A \neq 0° + k \cdot 90°, k\epsilon\mathbb{Z}$

And:

$\tan 2A \neq 0$

$\therefore 2A \neq 90° + k \cdot 180°$

$\therefore A \neq 45° + k \cdot 90°, k\epsilon\mathbb{Z}$

And for $\tan A$:

$A \neq 90° + k \cdot 180°, k\epsilon\mathbb{Z}$

3.2 $A = 216{,}87°$

4.1 $\sqrt{\frac{2-\sqrt{2}}{4}}$

4.2 $\sqrt{\frac{2-\sqrt{2}}{4}}$

4.3 $\sqrt{\frac{2+\sqrt{2}}{4}}$

5.1 $LHS = \tan 2x + \frac{1}{\cos 2x}$

$= \frac{\sin 2x}{\cos 2x} + \frac{1}{\cos 2x}$

$= \frac{\sin 2x + 1}{\cos 2x}$

$= \frac{2 \sin x \cos x + \cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x}$

$= \frac{(\cos x + \sin x)^2}{(\cos x - \sin x)(\cos x + \sin x)}$

$= \frac{\sin x + \cos x}{\cos x - \sin x}$

$= RHS$

Restrictions:

$\cos 2x \neq 0 \therefore 2x \neq 90° + k \cdot 180°$

$\therefore x \neq 45° + k \cdot 90°, k\epsilon\mathbb{Z}$

And:

$\cos x \neq \sin x$

$\therefore x \neq 45° + k \cdot 180°, k\epsilon\mathbb{Z}$

And for $\tan 2x$:

$2x \neq 90° + k \cdot 180°$

$\therefore x \neq 45° + k \cdot 90°, k\epsilon\mathbb{Z}\mathbb{Z}$

5.2 Consider the denominator on the LHS:

$\cos 2x$

$= \cos(2 \times 45°)$

$= \cos(90°)$

$= 0$

Consider the denominator on the RHS:

$\cos 45°$

$= \sin 45°$

$\therefore \cos 45° - \sin 45° = 0$

Therefore, the identity will be undefined because division by zero is not permitted.

8.4 Exercise 4

1.1 LHS $= \tan y$

$= \frac{\sin y}{\cos y}$

RHS $= \frac{\sin 2y}{\cos 2y+1}$

$= \frac{2\sin y \times \cos y}{\cos^2 y - 1 + 1}$

$= \frac{2\sin y \times \cos y}{\cos^2 y}$

$= \frac{\sin y}{\cos y}$

$\therefore$ LHS $=$ RHS

1.2 LHS $= \frac{\cos 2x}{\sin 2x+1}$

$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + 2\sin x \cos x + \sin^2 x}$

$= \frac{(\cos x - \sin x)(\cos x \sin x)}{(\cos x + \sin x)^2}$

$= \frac{\cos x - \sin x}{\cos x + \sin x}$

$=$ RHS

$\therefore$ LHS $=$ RHS

1.3 LHS

$= \tan A + \sin 2A$

$= \frac{\sin A}{\cos A} + 2\sin A \cos A$

$= \frac{\sin A + 2\sin A \cos^2 A}{\cos A}$

$= \frac{\sin A(1+2\cos^2 A)}{\cos A}$

RHS

$= \tan A(1 + 2\cos^2 A)$

$= \frac{\sin A}{\cos A}(1 + 2\cos^2 A)$

$= \frac{\sin A + 2\sin A \cos^2 A}{\cos A}$

$=$ RHS

$\therefore$ LHS $=$ RHS

1.4 LHS

$= \frac{1-2\cos^2 x}{\frac{1}{\tan 2x}}$

$= \frac{1-2\cos^2 x}{\frac{\cos 2x}{\sin 2x}}$

$= \frac{-(2]cos^2 x - 1}{\frac{\cos 2x}{\sin 2x}}$

$= \frac{-\cos 2x \sin 2x}{\cos 2x}$

$= -\sin 2x$

$= -2\sin x \cos x =$ RHS

$\therefore$ LHS $=$ RHS

1.5 LHS

$= \cos^2 2\theta - \sin^2 2\theta$

$= \cos^2 2\theta - 4\sin^2\theta\cos^2\theta$

$= 2\cos^2\theta - 1 - 4(1-\cos^2\theta)(\cos^2\theta)$

$= 2\cos^2\theta - 1 - 4(\cos^2\theta - \cos^4\theta)$

$= 2\cos^2\theta - 1 - 4\cos^2\theta + 4\cos^4\theta$

$= 4\cos^4\theta - 2\cos^2\theta - 1$

$=$ RHS

$\therefore$ LHS $=$ RHS

1.6 LHS

$= \frac{\frac{1}{\tan A}+1}{\tan A+1}$

$= \frac{\frac{\cos A}{\sin A}+1}{\frac{\sin A}{\cos A}+1}$

$= \frac{\frac{\cos A+\sin A}{\sin A}}{\frac{\sin A+\cos A}{\cos A}}$

$= \frac{(\cos A+\sin A)(\cos A)}{(\cos A+\sin A)(\sin A)}$

$= \frac{\cos A}{\sin A}$

$= \frac{1}{\tan A}$

$=$ RHS

$\therefore$ LHS $=$ RHS

1.7 LHS

$= \frac{1-\cos 4\beta}{\sin 2\beta}$

$= \frac{1-(1-2\sin^2\beta)}{2\sin\beta\cos\beta}$

$= \frac{2\sin^2\beta}{2\sin\beta\cos\beta}$

$= \frac{\sin\beta}{\cos\beta}$

$= \tan\beta$

$=$ RHS

$\therefore$ LHS $=$ RHS

1.8 RHS

$= (\cos x + \sin x)(\cos x + 3\sin x)$

$= \cos^2 x + 3\sin x\cos x + \sin x\cos x + 3\sin^2 x$

$= (\cos^2 x + \sin^2 x) + 4\sin x\cos x + 2\sin^2 x$

$= 1 + 2\sin 2x + 2\sin^2 x$

$=$ LHS

$\therefore$ RHS $=$ LHS

1.9 LHS

$= \frac{1}{1+\sin x} + \frac{1}{1-\sin x}$

$= \frac{1-\sin x+1+\sin x}{(1+\sin x)(1-\sin x)}$

$= \frac{2}{1-\sin^2 x}$
$= \frac{2}{\cos^2 x}$

RHS
$= \frac{\sin 2x}{\cos^4 x \tan x}$
$= \frac{2 \sin x \cos x}{\cos^4 x \frac{\sin x}{\cos x}}$
$= \frac{2}{\cos^2 x}$

$\therefore$ LHS $=$ RHS

2.1 $\cos 4y + 1 = 0$
$\cos 4y = -1$
REF$\angle = 0$

$2y = 180° + k\dot{3}60°$
$\therefore y = 90° + k\dot{1}80°, k\epsilon\mathbb{Z}$

or

$\therefore \tan y = 0$
$y = 90° + k \cdot 180°, k\epsilon\mathbb{Z}$

2.2 $\sin 2x + 1 = 0$
$\sin 2x = -1$
RA $= 90°$

Quartile 4:
$\therefore 2x = 360° - 90° + k \cdot 360°$
$2x = 270° + k \cdot 360°$
$x = 135° + k \cdot 180°, k\epsilon\mathbb{Z}$

OR:

Quartile 3:
$2x = 180° + 90° + k \cdot 360°$
$x = 135° + k \cdot 180°, k\epsilon\mathbb{Z}$

OR:

$\cos x + \sin x = 0$

$\cos x = -\sin x$

$-1 = \tan x$

$\text{RA} = 45°$

Quartile 2:

$x = 135° + k \cdot 180°, k\epsilon\mathbb{Z}$

OR:

$x = 315° + k \cdot 180°, k\epsilon\mathbb{Z}$

2.3 $A = 90° + k \cdot 180°, k\epsilon\mathbb{Z}$

2.4 $\frac{1}{\tan 2x} = 0$ OR $x = 90° + k \cdot 180°, k\epsilon\mathbb{Z}$

$\frac{\cos 2x}{\sin 2x} = 0 \therefore \cos 2x = 0\text{RA} = 90°$

Quartile 1:

$2x = 90° + k \cdot 360° x = 45° + k \cdot 180°, k\epsilon\mathbb{Z}$

OR

Quartile 4:

$2x = 360° - 90° + k \cdot 360°$

$x = 135° + k\dot{1}80°, k\epsilon\mathbb{Z}$

2.5 No Restrictions

2.6 $1 + \frac{1}{\tan A} = 0$ OR $A = 90° + k \cdot 180°, k\epsilon\mathbb{Z}$

$\frac{1}{\tan A} = -1$

$1 = -\tan A$

$\tan A = -1$

$\text{RA} = 45°$

Quartile 2:

$135° + k \cdot 180°, k\epsilon\mathbb{Z}$

OR:

Quartile 4:

$315° + k \cdot 180°, k\epsilon\mathbb{Z}$

2.7 $A = 90° + k \cdot 180°, k\epsilon\mathbb{Z}$

$\tan A = 0$
RA $= 0$

$A = k \cdot 180°, k\epsilon\mathbb{Z}$

$\tan A + 1 = 0$
$\tan A = -1$
RA $= 45°$

Quartile 2:
$A = 135° + k \cdot 180°, k\epsilon\mathbb{Z}$

Quartile 4:
$315° + k \cdot 180°, k\epsilon\mathbb{Z}$

2.8 $\beta = 90° + k \cdot 180°, k\epsilon\mathbb{Z}$
$\sin 2\beta = 0$
RA $= 0$

Quartile 2:
$2\beta = 0 + k \cdot 360°, k\epsilon\mathbb{Z}$
$\beta = k180°, k\epsilon\mathbb{Z}$

OR

Quartile 2:
$2\beta = 180° + k \cdot 360°, k\epsilon\mathbb{Z}$
$\beta = 90° + k180°, k\epsilon\mathbb{Z}$

2.9 No Restrictions

2.10 $1 + \sin x$
$\sin x = -1$
RA $= 90°$

$x = 270° + k \cdot 360°, k\epsilon\mathbb{Z}$

$1 - \sin x = 0$
$\sin x = 1$
RA $= 90°$

$x = 90° + k \cdot 360°, k\epsilon\mathbb{Z}$

$\cos^4 x \cdot \tan x = 0$
$\cos^3 x \cdot \sin x = 0$

$\therefore \cos^3 x = 0$
$\cos x = 0$
$\text{RA} = 90°$

$x = 90° + k \cdot 360°, k\epsilon\mathbb{Z}$
OR
$x = 270° + k \cdot 360°, k\epsilon\mathbb{Z}$

OR
$\sin x = 0$
$\text{RA} = 0$

$\therefore x = k \cdot 360°, k\epsilon\mathbb{Z}$
OR
$x = 180° + k \cdot 360°, k\epsilon\mathbb{Z}$

8.5 Exercise 5

1.1 $x = 16{,}06° + k \cdot 180°$ or $73{,}90° + k \cdot 180°$

1.2 $\alpha = 22{,}50° + k \cdot 90°, k\epsilon\mathbb{Z}$

1.3 First quadrant: $A = 45° + k \cdot 180°$
Second quadrant: $A = 135° + k \cdot 180°, k\epsilon\mathbb{Z}$
Restrictions: $A \neq 0° + k \cdot 180°, k\epsilon\mathbb{Z}$

1.4 $x = 78{,}19° + k \cdot 360°, k\epsilon\mathbb{Z}$ or $x = 341{,}81° + k \cdot 360°, k\epsilon\mathbb{Z}$

1.5 First quadrant: $x = 51{,}8° + k \cdot 360°$
Second quadrant: $x = (360° - 51{,}8°) + k \cdot 360°$
Restrictions: $x \neq 90° + k \cdot 180°, k\epsilon\mathbb{Z}$

1.6 $x = 60° + k \cdot 360°$

2.1 First quadrant: $y = 30° + k \cdot 120°$
Second quadrant: $y = 90° + k \cdot 360°, k\epsilon\mathbb{Z}$

2.2 First quadrant: $p = k \cdot 360°$

Second quadrant: $p = 36° + k \cdot 72°, k\epsilon\mathbb{Z}$

2.3 $x = 120° + k \cdot 360°$ or $x = 26{,}67° + k \cdot 360°; k\epsilon\mathbb{Z}$

2.4 $A = 60° + k \cdot 60°$

2.5 $\beta = 335° + k \cdot 320°$ OR $\beta = 128{,}33° + k \cdot 120°; k\epsilon\mathbb{Z}$

2.6 $x = 80° + k \cdot 120°$ or $x = 120° + k \cdot 360°; k\epsilon\mathbb{Z}$

3.1 $t = 0° + k \cdot 180°, k\epsilon\mathbb{Z}$

3.2 $x = 30° + k \cdot 360°$

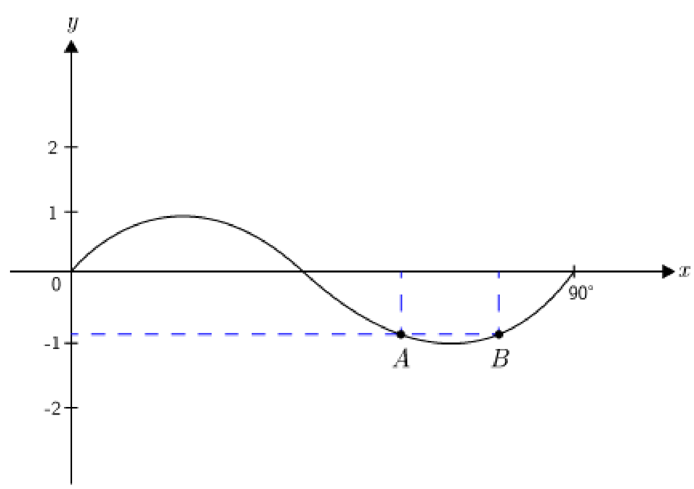
3.3 $A = 21{,}68° + k \cdot 360°$ or $A = 278{,}32° + k \cdot 360°, k\epsilon\mathbb{Z}$

4. $A = -300°, -60°, 60°$or$300°$

5.1 Third quadrant: $59{,}26° + k \cdot 90°$

Fourth quadrant: $x = 75{,}74° + k \cdot 90°$

5.2 The diagram below shows the graph of $y = \sin 4x$



$A(59{,}26; -0{,}84), B(75{,}74l; -0{,}84)$

6. $\sin(x + 60°) = -\frac{1}{2}\sin x$

$\sin x \cos 60° + \cos x \sin 60° = -\frac{1}{2}\sin x$

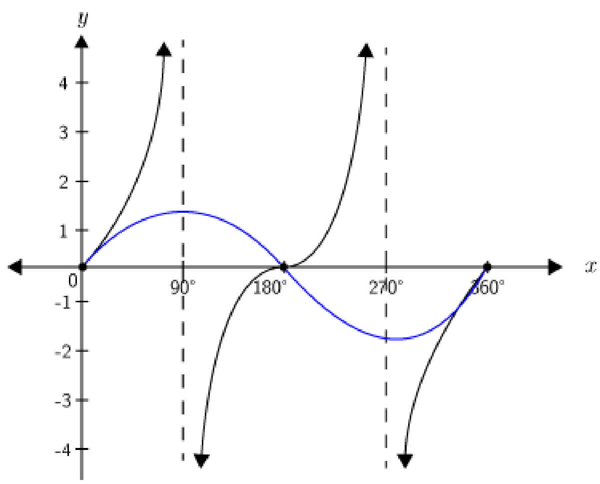
$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = -\frac{1}{2}\sin x$

$\frac{\sqrt{3}}{2}\cos x = -\frac{1}{2}\sin x - \frac{1}{2}\sin x$

$\frac{\sqrt{3}}{2} \cos x = -\sin x$
$\sin x = -\frac{\sqrt{3}}{2} \cos x$
$\tan x = \frac{\sin x}{\cos x}$
$\tan x = \frac{-\frac{\sqrt{3}}{2} \cos x}{\cos x}$
$\tan x = -\frac{\sqrt{3}}{2}$

7.1 $x = 30° \text{or} 210°$

7.2 The diagram below shows the graph indicating the solution



8.1 $A = 35{,}79° + k \cdot 90°, k\epsilon\mathbb{Z}$ or $A = 31{,}72° + k \cdot 90°, k\epsilon\mathbb{Z}$

8.2 There are ten solutions that lie within the interval $[-90°; 360°]$

8.6 Exercise 6

1.1 2θ

1.2 $\cos\theta = \frac{AE}{BE}$

1.3 $\sin\theta = \frac{CA}{AE}$

1.4 $\sin 2\theta = \frac{CA}{AO}$

1.5 $2\sin\theta\cos\theta$

2. $\frac{1}{4}$

3.1 $DC \cdot DA \cdot \sin \hat{D}$

3.2 $CA^2 = DC^2 + DA^2 – 2(DC)(DA) \cos \hat{D}$
$CA^2 = AB^2 + BC^2 – 2(AB)(BC) \cos \hat{B}$

3.3 $CD^2 + DA^2 + AB^2 + BC^2$

3.4 $188{,}5$ units^2

3.5 $\hat{B} = 115{,}94°$
Area $= 54{,}0$ units^2

4.1 64 m

4.2 64 m

8.7 Exercise 7

1.1 $BC = 2x \cos \alpha \tan \theta$

1.2 $BC = 2(140) \cos 21° \tan 9° = 41{,}40$ m

2.1 $h = \frac{d \sin x \tan z}{\sin(x+y)}$

2.2 $h = 15$ m

3.1 QR is the link between ΔPQR and ΔSQR
In ΔSQR: $\hat{S} = 180° - (120° + \alpha)(\angle$'s sum of ΔSQR

$\frac{QR}{\sin \hat{S}} = \frac{q}{\sin \hat{Q}}$

$QR = \frac{q \sin(180° - (120° + \alpha))}{\sin(120°)}$

$= \frac{q \sin(60° - a)}{\sin(180° - 60°)}$

$= \frac{q(\sin 60° \cos \alpha - \cos 60° \sin \alpha)}{\sin 60°}$

$= \frac{q\left(\frac{\sqrt{3}}{2} \cos \alpha - \frac{1}{2} \sin \alpha\right)}{\frac{\sqrt{3}}{2}}$

$= q\left(\frac{\sqrt{3} \cos \alpha - \sin \alpha}{2}\right) \times \frac{2}{\sqrt{3}}$

$= q\left(\cos \alpha - \frac{\sin \alpha}{\sqrt{3}}\right)$

$= q\left(\cos \alpha - \left(\frac{\sin \alpha}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right)\right)$

$= q\left(\cos \alpha - \frac{\sqrt{3} \sin \alpha}{3}\right)$

In $\triangle PQR : \hat{R} = 90°$

$\frac{PR}{QR} = \tan \theta$

$\therefore PR = \tan \theta \times QR$

$= \tan \theta \times q\left(\cos \alpha - \frac{\sqrt{3} \sin \alpha}{3}\right)$

$= q \tan \theta \left(\cos \alpha - \frac{\sqrt{3} \sin \alpha}{3}\right)$

3.2 $PR = 16{,}2$ m

3.3 6 levels

4.1 $400 \cos \beta$

4.2 $\Delta LSB = 2\ 000 \cos^2 \alpha \sin \theta$

4.3 $1\ 932{,}3$ m

5.1 229 m^2

5.2 $1 - \frac{x^2}{2d^2}$

5.3 $148°$