CHAPTER 5

Polynomials

MathU Teaching® Emporium

CONTENTS

August 26, 2021

1 REVISION

1.1 Identifying polynomials

Terminology	
Polynomial	An expression that involves one or more variables having different powers and coefficients. $a_nx^n + \ldots + a_2x^2 + a_1x + a_0,$ where $n \in \mathbb{N}_0$
Monomial	A polynomial with one term. For example, $7a^2b$ or $15xyz^2$
Binomial	A polynomial that has two terms. For example, $2x + 5z$ or $26 - g^2k$
Trinomial	A polynomial that has three terms. For example, $a - b + c$ or $4x^2 + 17xy - y^3$
Degree/Order	The degree, also called the order, of a univariate polynomial is the value of the highest exponent in the polynomial. For example, $7p - 12p^2 + 3p^5 + 8$ has a degree of **5**

It is important to notice that the definition of a polynomial states that all exponents of the variables must be elements of the set of natural numbers. If an expression contains terms with exponents that are not natural numbers, then it is not a polynomial.

The following examples are not polynomials:

$$\frac{3}{y} - 4y^2 + 1$$
$$5\sqrt{k} + k - 2k^2$$
$$x^{-2} + 3 + 7x^2$$
$$t^2 - 4t + 6t^{\frac{1}{3}}$$

INVESTIGATION

More on polynomials

Discuss whether the following statements are true or false:

1. The expression $3y^2 + 2y – 4$ is a trinomial of degree 2.
2. $25z^5 - 36\sqrt{z}$ is a binomial of order 5.
3. 25 is a constant polynomial of degree 0.
4. $3x^2 – 2x – 5$ is a quadratic polynomial.
5. The expression $23b^{–2}$ is a monomial because it only has one term.
6. 0 is a constant polynomial of undefined degree.
7. A cubic polynomial has three terms and all the exponents are natural numbers.
8. Given the expression : $\frac{1}{t} - 3t^2 + 1$
   If we multiply by $t : 1 - 3t^3 + t$,
   we get a trinomial of degree 3.

1.2 Quadratic polynomials

In earlier grades we learnt the following useful techniques and methods for factorising an expression:

- taking out a common factor
- factorising the difference of two squares
- grouping in pairs
- factorising the sum and difference of two cubes

We also looked at the different methods for factorising quadratic expressions:

- factorising by inspection
- completing the square
- using the quadratic formula
- making a suitable substitution

It is important to revise these methods; we use the quadratic formula for factorising cubic polynomials and we also use completing the square to find the equation of a circle in Chapter 7.

Terminology	
Variable	A symbol used to represent an unknown numerical value. For example, $a, b, x, y, \alpha, \theta$
Coefficient	The number or parameter that is multiplied by the variable of an expression.
Expression	A term or group of terms consisting of numbers, variables and the basic operators $(+, -, \times, \div)$
Univariate expression	An expression containing only one variable.
Equation	A mathematical statement that asserts that two expressions are equal.
Identity	A mathematical relationship that equates one expression with another.
Solution	A value or set of values that satisfies the original problem statement.
Root/Zero	A root, also referred to as the "zero", of an equation is the value of x such that $f(x) = 0$ is satisfied.

WORKED EXAMPLE 1: SOLVING QUADRATIC EQUATIONS USING FACTORISATION

QUESTION

Solve for x:

$$\frac{3x}{x+2} + 1 = \frac{4}{x+1}$$

SOLUTION

Step 1: Determine the restrictions

The restrictions are the values for x that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $x \neq -2$ and $x \neq -1$.

Step 2: Determine the lowest common denominator

The lowest common denominator is $(x+2)(x+1)$.

Step 3: Multiply each term in the equation by the lowest common denominator and simplify

$$\frac{3x(x+2)(x+1)}{x+2} + (x+2)(x+1) = \frac{4(x+2)(x+1)}{x+1}$$

$$3x(x+1) + (x+2)(x+1) = 4(x+2)$$

$$3x^2 + 3x + x^2 + 3x + 2 = 4x + 8$$

$$4x^2 + 2x - 6 = 0$$

$$2x^2 + x - 3 = 0$$

Step 4: Factorise and solve the equation

$$(2x+3)(x-1) = 0$$

$$2x + 3 = 0 \text{ or } x - 1 = 0$$

$$x = -\frac{3}{2} \text{ or } x = 1$$

Step 5: Check the solution by substituting both answers back into the original equation

Step 6: Write the final answer

Therefore $x = -1\frac{1}{2}$ or $x = 1$.

WORKED EXAMPLE 2: USING THE QUADRATIC FORMULA

QUESTION

Find the roots of the function $f(x) = 3x^2 + 4x-8$.

SOLUTION

Step 1: Finding the roots

To determine the roots of $f(x)$, we let $3x^2 + 4x-8 = 0$.

Step 2: Check whether the expression can be factorised

The expression cannot be factorised, so the general quadratic formula must be used.

Step 3: Identify the coefficients to substitute into the formula

$$a = 3; \qquad b = 4; \qquad c = -8$$

Step 4: Apply the quadratic formula

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(4) \pm \sqrt{(4)^2 - 4(3)(-8)}}{2(3)} \\ &= \frac{-4 \pm \sqrt{16 + 96}}{6} \\ &= \frac{-4 \pm \sqrt{112}}{6} \\ &= \frac{-4 \pm \sqrt{16 \times 7}}{6} \\ &= \frac{-4 \pm 4\sqrt{7}}{6} \\ &= \frac{-2 \pm 2\sqrt{7}}{3} \end{aligned}$$

Step 5: Write the final answer

Therefore $x = \dfrac{-2 + 2\sqrt{7}}{3}$ or $x = \dfrac{-2 - 2\sqrt{7}}{3}$

WORKED EXAMPLE 3: SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

QUESTION

Solve by completing the square: $y^2–10y–11 = 0$

SOLUTION

Step 1: The equation is already in the form $ax^2 + bx + c = 0$

Step 2: Make sure the coefficient of the y^2 **term is equal to** 1

$$y^2 - 10y - 11 = 0$$

Step 3: Take half the coefficient of the y **term and square it, then add and subtract it from the equation**

The coefficient of the y term is $–10$. Half of the coefficient of the y term is $–5$ and the square of it is 25. Therefore $y^2–10y + 25–25–11 = 0$.

Step 4: Write the trinomial as a perfect square

$$(y^2 - 10y + 25) - 25 - 11 = 0$$

$$(y - 5)^2 - 36 = 0$$

Step 5: Method 1: Take square roots on both sides of the equation

$$(y - 5)^2 = 36$$

$$y - 5 = \pm\sqrt{36}$$

Important: When taking a square root always remember that there is a positive and negative answer, since $(6)^2 = 36$ and $(–6)^2 = 36$.

$$y - 5 = \pm 6$$

Step 6: Solve for y

If $y - 5 = 6$

$$y = 11$$

Or if $y - 5 = -6$

$$y = -1$$

Therefore $y = 11$ or $y = –1$.

WORKED EXAMPLE 3: SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE CONTINUES

Step 7: Method 2: Factorise the expression as a difference of two squares

$$(y-5)^2 - (6)^2 = 0$$
$$[(y-5)+6]\,[(y-5)-6] = 0$$

Step 8: Simplify and solve for y

$$(y+1)(y-11) = 0$$
$$\therefore y = -1 \text{ or } y = 11$$

Step 9: Write the final answer

$$y = -1 \text{ or } y = 11$$

Notice that both methods produce the same answer. These roots are rational because **36** is a perfect square.

2 CUBIC POLYNOMIALS

INVESTIGATION

Simple division

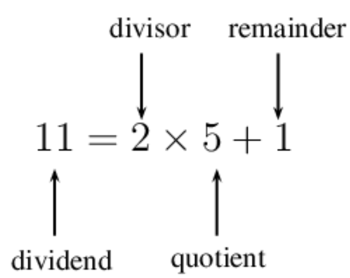
Consider the following and answer the questions below:

1. 6 students are at a product promotion and there are 15 free gifts to be given away. Each student must receive the same number of gifts.
   (a) Determine how many gifts each student will get?
   (b) How many gifts will be left over?
   (c) Use the following variables to express the above situation as a mathematical equation:
      - $a =$ total number of gifts
      - $b =$ total number of students
      - $q =$ number of gifts for each student
      - $r =$ number of gifts left over
2. A group of students go to dinner together at a restaurant and the total bill is R510. Each student contributes R120 towards the bill. They count the money and find that they are still R30 short.
   (a) Assign variables to the known and unknown values.
   (b) Write a mathematical equation to describe the situation.
   (c) Use this equation to determine how many students went to dinner.

We know that 11 divided by 2 gives an answer of 5 with a remainder 1.

$$\frac{11}{2} = 5 \text{ remainder } 1$$
$$\frac{11}{2} = 5 + \frac{1}{2}$$

This means that:



We can write a general expression for the rule of division; if an integer a is divided by an integer b, then the answer is q with a remainder r.

$$\frac{a}{b} = q + \frac{r}{b}$$
$$a = b \times q + r$$

where $b \neq 0$ and $0 \leq r < b$.

This rule can be extended to include the division of polynomials; if a polynomial $a(x)$ is divided by a polynomial $b(x)$, then the answer is $Q(x)$ with a remainder $R(x)$.

$$a(x) = b(x) \times Q(x) + R(x)$$

where $b(x) \neq 0$.

In words: the dividend is equal to the divisor multiplied by the quotient, plus the remainder.

A cubic polynomial is an expression with the highest power equal to 3; we say that the degree of the polynomial is 3.

The general form of a cubic polynomial is

$$y = ax^3 + bx^2 + cx + d$$

where $a \neq 0$.

In Grade 10 we learnt how to factorise the sum and difference of two cubes by first finding a factor (the first bracket) and then using inspection (the second bracket). For example,

$$p^3 + 8 = (p + 2)(p^2 - 2p + 4)$$
$$k^3 - 1 = (k - 1)(k^2 + k + 1)$$

In this section we focus on factorising cubic polynomials with one variable (univariate) where b and c are not zero.

We use the following methods for factorising cubic polynomials:

- long division
- synthetic division
- inspection

WORKED WORKED EXAMPLE 4: LONG DIVISION

QUESTION

Use the long division method to determine the quotient $Q(x)$ and the remainder $R(x)$ if $a(x) = 2x^3 – x^2 – 6x + 16$ is divided by $b(x) = x – 1$. Write your answer in the form $a(x) = b(x)\,?\,Q(x) + R(x)$.

SOLUTION

Step 1: Write down the known and unknown expressions

$$a(x) = b(x) \cdot Q(x) + R(x)$$
$$2x^3 - x^2 - 6x + 16 = (x - 1) \cdot Q(x) + R(x)$$

Step 2: Use long division method to determine $Q(x)$ and $R(x)$

Make sure that $a(x)$ and $b(x)$ are written in descending order of the exponents. If a term of a certain degree is missing from $a(x)$, then write the term with a coefficient of 0.

$$\begin{array}{r} 2x^2 + x - 5 \\ x - 1|\overline{2x^3 - x^2 - 6x + 16} \\ -\left(\underline{2x^3 - 2x^2}\right) \\ 0 + x^2 - 6x \\ -\left(\underline{x^2 - x}\right) \\ 0 - 5x + 16 \\ -\left(\underline{-5x + 5}\right) \\ 0 + 11 \end{array}$$

Step 3: Write the final answer

$$Q(x) = 2x^2 + x - 5$$
$$R(x) = 11$$
$$\text{and } a(x) = b(x) \cdot Q(x) + R(x)$$
$$\therefore a(x) = (x - 1)(2x^2 + x - 5) + 11$$

Synthetic division is a simpler and more efficient method for dividing polynomials. It allows us to determine the quotient and the remainder by considering the coefficients of the terms in each of the polynomials without needing to rewrite the variable and exponent for each term. If a term of a certain degree is missing from $a(x)$, then write the term with a coefficient of 0. For example, $a(x) = 5x^3 + 6x – 1$ should be written as $a(x) = 5x^3 + 0x^2 + 6x – 1$.

Notice that for synthetic division:

- the coefficients of the dividend ($a(x)$) are written below the horizontal line.
- the coefficients of the quotient ($Q(x)$) are written above the horizontal line.
- we add coefficients instead of subtracting as is the case with long division.
- we use the opposite sign of the divisor ($b(x)$); the divisor is (x–1) and we use $+1$.
- the coefficient of the x term in the divisor is 1, so $q_2 = a_3$.

WORKED WORKED EXAMPLE 5: SYNTHETIC DIVISION

QUESTION

Use the synthetic division method to determine the quotient $Q(x)$ and the remainder $R(x)$ if $a(x) = 2x^3$–x^2–$6x$+16 is divided by $b(x) = x$–1. Write your answer in the form $a(x) = b(x)$□$Q(x)$+$R(x)$.

SOLUTION

Step 1: Write down the known and unknown expressions

$$a(x) = b(x) \cdot Q(x) + R(x)$$
$$2x^3 - x^2 - 6x + 16 = (x - 1) \cdot Q(x) + R(x)$$

Step 2: Use synthetic division to determine $Q(x)$ and $R(x)$

$$\begin{array}{r|rrrr} & 2 & +1 & -5 & 11 \\ \hline 1 & 2 & -1 & -6 & 16 \end{array}$$

$$\begin{aligned} q_2 &= 2 \\ q_1 &= -1 + (2)(1) = 1 \\ q_0 &= -6 + (1)(1) = -5 \\ R &= 16 + (-5)(1) = 11 \end{aligned}$$

Step 3: Write the final answer

The quotient will be one degree lower than the dividend if we divide by a linear expression, therefore we have:

$$Q(x) = 2x^2 + x - 5$$
$$R(x) = 11$$
and $a(x) = b(x) \cdot Q(x) + R(x)$
$$\therefore a(x) = (x - 1)(2x^2 + x - 5) + 11$$

General method: Given the dividend $a(x) = a_3x^3 + a_2x^2 + a_1x^1 + a_0x^0$ and the divisor $(cx – d)$, we determine the quotient $Q(x) = q_2x^2 + q_1x^1 + q_0x^0$ and the remainder $R(x)$ using:

$$\begin{array}{c|cccc} & q_2 & q_1 & q_0 & R \\ \hline \frac{d}{c} & a_3 & a_2 & a_1 & a_0 \end{array}$$

We determine the coefficients of the quotient by calculating:

$$\begin{aligned} q_2 &= a_3 + \left(q_3 \times \frac{d}{c}\right) \\ &= a_3 \quad \text{(since } q_3 = 0) \\ q_1 &= a_2 + \left(q_2 \times \frac{d}{c}\right) \\ q_0 &= a_1 + \left(q_1 \times \frac{d}{c}\right) \\ R &= a_0 + \left(q_0 \times \frac{d}{c}\right) \end{aligned}$$

Important note: $a(x)$ is a function and a_3, a_2, a_1 and a_0 are coefficients.

WORKED WORKED EXAMPLE 6: SYNTHETIC DIVISION

QUESTION

Use the synthetic division method to determine the quotient $Q(x)$ and the remainder $R(x)$ if $a(x) = 6x^3 - x^2 - 4x + 5$ is divided by $b(x) = 2x - 1$.

SOLUTION

Step 1: Write down the known and unknown expressions

$$a(x) = b(x) \cdot Q(x) + R(x)$$
$$6x^3 + x^2 - 4x + 5 = (2x - 1) \cdot Q(x) + R(x)$$

Step 2: Use synthetic division to determine $Q(x)$ and $R(x)$

Make the leading coefficient of the divisor equal to 1:

$$b(x) = (2x - 1) = 2\left(x - \frac{1}{2}\right)$$

$$\begin{array}{c|cccc} & 6 & 4 & -2 & 4 \\ \hline \frac{1}{2} & 6 & 1 & -4 & 5 \end{array}$$

$$\begin{aligned} q_2 &= 6 \\ q_1 &= 1 + (6)\left(\frac{1}{2}\right) = 4 \\ q_0 &= -4 + (4)\left(\frac{1}{2}\right) = -2 \\ R &= 5 + (-2)\left(\frac{1}{2}\right) = 4 \end{aligned}$$

Step 3: Write the final answer

$$\begin{aligned} Q(x) &= 6x^2 + 4x - 2 \\ &= 2\left(3x^2 + 2x - 1\right) \\ R &= 4 \\ \text{and } a(x) &= \frac{1}{2}b(x) \cdot Q(x) + R(x) \\ \therefore a(x) &= \frac{1}{2} \cdot 2\left(x - \frac{1}{2}\right)(6x^2 + 4x - 2) + 4 \\ &= \frac{1}{2} \cdot 2\left(x - \frac{1}{2}\right)(2)(3x^2 + 2x - 1) + 4 \\ &= (2x - 1)(3x^2 + 2x - 1) + 4 \end{aligned}$$

3 REMAINDER THEOREM

INVESTIGATION

Remainder theorem

Given the following functions:

- $f(x) = x^3 + 3x^2 + 4x + 12$
- $k(x) = x - 1$
- $g(x) = 4x^3 - 2x^2 + x - 7$
- $h(x) = x + 2$

1. Determine $\frac{f(x)}{k(x)}$ and $\frac{g(x)}{h(x)}$.
2. Write your answers in the general form: $a(x) = b(x).Q(x) + R(x)$.
3. Determine $f(1)$ and $g(-2)$.
4. What do you notice?
5. Consider the degree of the quotient and the remainder - is there a rule?
6. What conclusions can you draw?
7. Write a mathematical equation to describe your conclusions.
8. Complete the following sentence: a cubic function divided by a linear polynomial gives a quotient with a degree of . . . and a remainder with a degree of . . ., which is called a constant.

The Remainder theorem

A polynomial $p(x)$ divided by $cx-d$ gives a remainder of $p(\frac{d}{c})$.

In words: the value of the remainder R is obtained by substituting $x = \frac{d}{c}$ into the polynomial $p(x)$.

$$R = p\left(\frac{d}{c}\right)$$

NOTE: PROOF NOT FOR EXAMS Let the quotient be $Q(x)$ and let the remainder be R. Therefore we can write:

$$\begin{aligned}
p(x) &= (cx - d) \cdot Q(x) + R \\
\therefore p\left(\frac{d}{c}\right) &= \left[c\left(\frac{d}{c}\right) - d\right] \cdot Q\left(\frac{d}{c}\right) + R \\
&= (d - d) \cdot Q\left(\frac{d}{c}\right) + R \\
&= 0 \cdot Q\left(\frac{d}{c}\right) + R \\
&= R \\
\therefore p\left(\frac{d}{c}\right) &= R
\end{aligned}$$

WORKED EXAMPLE 7: FINDING THE REMAINDER

QUESTION

Use the remainder theorem to determine the remainder when $p(x) = 3x^3 + 5x^2 - x + 1$ is divided by the following linear polynomials:

1. $x + 2$
2. $2x - 1$
3. $x + m$

SOLUTION

Step 1: Determine the remainder for each linear divisor

The remainder theorem states that any polynomial $p(x)$ that is divided by $cx - d$ gives a remainder of $p(\frac{d}{c})$:

1.

$$\begin{aligned} p(x) &= 3x^3 + 5x^2 - x + 1 \\ p(-2) &= 3(-2)^3 + 5(-2)^2 - (-2) + 1 \\ &= 3(-8) + 5(4) + 2 + 1 \\ &= -24 + 20 + 3 \\ \therefore R &= -1 \end{aligned}$$

2.

$$\begin{aligned} p(x) &= 3x^3 + 5x^2 - x + 1 \\ p\left(\frac{1}{2}\right) &= 3\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 1 \\ &= 3\left(\frac{1}{8}\right) + 5\left(\frac{1}{4}\right) - \left(\frac{1}{2}\right) + 1 \\ &= \frac{3}{8} + \frac{5}{4} + \frac{1}{2} \\ &= \frac{3}{8} + \frac{10}{8} + \frac{4}{8} \\ \therefore R &= \frac{17}{8} \end{aligned}$$

3.

$$\begin{aligned} p(x) &= 3x^3 + 5x^2 - x + 1 \\ p(m) &= 3(-m)^3 + 5(-m)^2 - (-m) + 1 \\ \therefore R &= -3m^3 + 5m^2 + m + 1 \end{aligned}$$

WORKED EXAMPLE 8: USING THE REMAINDER TO SOLVE FOR AN UNKNOWN VARIABLE

QUESTION

Given that $f(x) = 2x^3 + x^2 + kx + 5$ divided by $2x–3$ gives a remainder of $9\frac{1}{2}$, use the remainder theorem to determine the value of k.

SOLUTION

Step 1: Use the remainder theorem to determine the unknown variable k

From the remainder theorem we know that $f\left(\frac{3}{2}\right) = 9\frac{1}{2}$ and we can therefore solve for k:

$$\begin{aligned}
f(x) &= 2x^3 + x^2 + kx + 5 \\
f\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 + k\left(\frac{3}{2}\right) + 5 \\
9\frac{1}{2} &= 2\left(\frac{27}{8}\right) + \left(\frac{9}{4}\right) + k\left(\frac{3}{2}\right) + 5 \\
9\frac{1}{2} &= \frac{27}{4} + \frac{9}{4} + \frac{3k}{2} + 5 \\
9\frac{1}{2} &= \frac{36}{4} + \frac{3k}{2} + 5 \\
\therefore 9\frac{1}{2} - 9 - 5 &= \frac{3k}{2} \\
-4\frac{1}{2} &= \frac{3k}{2} \\
-\frac{9}{2} \times \frac{2}{3} &= k \\
\therefore -3 &= k
\end{aligned}$$

Step 2: Write the final answer

Therefore $k = –3$ and $f(x) = 2x^3 + x^2–3x + 5$.

4 FACTOR THEOREM

If an integer a is divided by an integer b, and the answer is q with the remainder $r = 0$, then we know that b is a factor of a.

$$a = b \times q + r$$
$$\text{and } r = 0$$
$$\text{then we know that } a = b \times q$$
$$\text{and also that } \frac{a}{b} = q$$

This is also true of polynomials; if a polynomial $a(x)$ is divided by a polynomial $b(x)$, and the answer is $Q(x)$ with the remainder $R(x) = 0$, then we know that $b(x)$ is a factor of $a(x)$.

$$a(x) = b(x) \cdot Q(x) + R(x)$$
$$\text{and } R(x) = 0$$
$$\text{then we know that } a(x) = b(x) \cdot Q(x)$$
$$\text{and also that } \frac{a(x)}{b(x)} = Q(x)$$

The factor theorem describes the relationship between the root of a polynomial and a factor of the polynomial.

The Factor theorem

If the polynomial $p(x)$ is divided by $cx–d$ and the remainder, given by $p(dc)$, is equal to zero, then $cx–d$ is a factor of $p(x)$.

Converse: if $(cx–d)$ is a factor of $p(x)$, then $p(\frac{d}{c}) = 0$.

WORKED EXAMPLE 9: FACTOR THEOREM

QUESTION

Using the factor theorem, show that $y+4$ is a factor of $g\left(y\right)=5y^4+16y^3-15y^2+8y+16$

SOLUTION

Step 1: Determine how to approach the problem

For $y+4$ to be a factor, $g(-4)$ must be equal to 0.

Step 2: Calculate $g(-4)$

$$\begin{aligned} g\left(y\right) &= 5y^4+16y^3-15y^2+8y+16 \\ \therefore g\left(-4\right) &= 5(-4)^4+16(-4)^3-15(-4)^2+8\left(-4\right)+16 \\ &= 5\left(256\right)+16\left(-64\right)-15\left(16\right)+8\left(-4\right)+16 \\ &= 1\ 280-1\ 024-240-32+16 \\ &= 0 \end{aligned}$$

Step 3: Conclusion

Since $g(-4)=0$, $y+4$ is a factor of $g(y)$.

In general, to factorise a cubic polynomial we need to do the following:

- Find one factor by trial and error: consider the coefficients of the given cubic polynomial $p(x)$ and guess a possible root $(\frac{c}{d})$.
- Use the factor theorem to confirm that $\frac{c}{d}$ is a root; show that $p(\frac{c}{d})=0$.
- Divide $p(x)$ by the factor $(cx-d)$ to obtain a quadratic polynomial (remember to be careful with the signs).
- Apply the standard methods of factorisation to determine the two factors of the quadratic polynomial.

WORKED EXAMPLE 10: FACTOR THEOREM

QUESTION

Using the factor theorem, show that $y-1$ is a factor of $f\left(y\right)=2y^4+3y^2-5y+7$

SOLUTION

Step 1: Determine how to approach the problem

For $y-1$ to be a factor, $f(1)$ must be equal to 0.

Step 2: Calculate $f(1)$

$$\begin{aligned} f\left(y\right) &= 2y^4+3y^2-5y+7 \\ \therefore f\left(1\right) &= 2(1)^4+3(1)^2-5\left(1\right)+7 \\ &= 2+3-5+7 \\ &= 7 \end{aligned}$$

Step 3: Conclusion

Since $f\left(1\right)\neq 0$, $y-1$ is not a factor of $f\left(y\right)=2y^4+3y^2-5y+7$.

WORKED EXAMPLE 11: FACTORISING CUBIC POLYNOMIALS

QUESTION

Factorise completely: $f(x) = x^3 + x^2 – 9x – 9$

SOLUTION

Step 1: Find a factor by trial and error

Try

$f(1) = (1)^3 + (1)^2 - 9(1) - 9 = 1 + 1 - 9 - 9 = -16$

Therefore $(x – 1)$ is not a factor.

We consider the coefficients of the given polynomial and try:

$f(-1) = (-1)^3 + (-1)^2 - 9(-1) - 9 = -1 + 1 + 9 - 9 = 0$

Therefore $(x + 1)$ is a factor, because $f(–1) = 0$.

Step 2: Factorise by inspection

Now divide $f(x)$ by $(x + 1)$ using inspection:

Write $x^3 + x^2 - 9x - 9 = (x + 1)(\ldots)$

The first term in the second bracket must be x^2 to give x^3 and make the polynomial a cubic.

The last term in the second bracket must be $–9$ because $(+1)(–9) = –9$.

So we have $x^3 + x^2 – 9x – 9 = (x + 1)(x^2 + ?x – 9)$

Now, we must find the coefficient of the middle term:

$(+1)(x^2)$ gives the x^2 in the original polynomial. So, the coefficient of the x-term must be 0.

$\therefore f(x) = (x + 1)(x^2 – 9)$. **Step 3: Write the final answer**

We can factorise the last bracket as a difference of two squares:

$$\begin{aligned} f(x) &= (x + 1)\left(x^2 - 9\right) \\ &= (x + 1)(x - 3)(x + 3) \end{aligned}$$

WORKED EXAMPLE 12: FACTORISING CUBIC POLYNOMIALS

QUESTION

Use the factor theorem to factorise $f(x) = x^3 - 2x^2 – 5x + 6$

SOLUTION

Step 1: Find a factor by trial and error

Try

$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 1 - 2 - 5 + 6 = 0$

Therefore $(x – 1)$ is a factor.

Step 2: Factorise by inspection

$x^3 - 2x^2 - 5x + 6 = (x - 1)(\ldots)$

The first term in the second bracket must be x^2 to give x^3 if we work backwards.

The last term in the second bracket must be $–6$ because $(-1)(–6) = +6$.

So we have $x^3 - 2x^2 - 5x + 6 = (x - 1)\left(x^2 + ?x - 6\right)$

Now, we must find the coefficient of the middle term:

$(-1)(x^2)$ gives the $-x^2$ in the original polynomial. So, the coefficient of the x-term must be -1 to give another $–x^2$ so that overall we have $–x^2 – x^2 = –2x^2$.

So $f(x) = (x - 1)\left(x^2 - x - 6\right)$

Make sure that the expression has been factorised correctly by checking that the coefficient of the x-term also works out: $(x)(–6) + (–1)(–x) = –5x$, which is correct.

Step 3: Write the final answer

We can factorise the last bracket as:

$$\begin{aligned} f(x) &= (x - 1)\left(x^2 - x - 6\right) \\ &= (x - 1)(x - 3)(x + 2) \end{aligned}$$

5 SOLVING CUBIC EQUATIONS

Now that we know how to factorise cubic polynomials, it is also easy to solve cubic equations of the form $ax^3 + bx^2 + cx + d = 0$.

WORKED EXAMPLE 13: SOLVING CUBIC EQUATIONS

QUESTION

Solve: $6x^3 - 5x^2 - 17x + 6 = 0$

SOLUTION

Step 1: Find one factor using the factor theorem

Let $f(x) = 6x^3 - 5x^2 - 17x + 6$

Try

$f(1) = 6(1)^3 - 5(1)^2 - 17(1) + 6 = 6 - 5 - 17 + 6 = -10$

Therefore $(x – 1)$ is not a factor.

Try

$f(2) = 6(2)^3 - 5(2)^2 - 17(2) + 6 = 48 - 20 - 34 + 6 = 0$

Therefore $(x - 2)$ is a factor.

Step 2: Factorise by inspection

$$6x^3 - 5x^2 - 17x + 6 = (x - 2)\left(6x^2 + 7x - 3\right)$$

Step 3: Factorise fully

$$6x^3 - 5x^2 - 17x + 6 = (x - 2)(2x + 3)(3x - 1)$$

Step 4: Solve the equation

$$6x^3 - 5x^2 - 17x + 6 = 0$$
$$(x - 2)(2x + 3)(3x - 1) = 0$$
$$x = 2 \text{ or } x = \frac{1}{3} \text{ or } x = -\frac{3}{2}$$

Sometimes it is not possible to factorise a quadratic expression using inspection, in which case we use the quadratic formula to fully factorise and solve the cubic equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

WORKED EXAMPLE 14: SOLVING CUBIC EQUATIONS

QUESTION

Solve for x: $0 = x^3 - 2x^2 - 6x + 4$

SOLUTION

Step 1: Find one factor using the factor theorem

Let $f(x) = x^3 - 2x^2 - 6x + 4$

Try

$f(1) = (1)^3 - 2(1)^2 - 6(1) + 4 = 1 - 2 - 6 + 4 = -3$

Therefore $(x–1)$ is not a factor.

Try

$f(2) = (2)^3 - 2(2)^2 - 6(2) + 4 = 8 - 8 - 12 + 4 = -8$

Therefore $(x - 2)$ is not a factor.

Try

$f(-2) = (-2)^3 - 2(-2)^2 - 6(-2) + 4 = -8 - 8 + 12 + 4 = 0$

Therefore $(x + 2)$ is not a factor.

Step 2: Factorise by inspection

$$x^3 - 2x^2 - 6x + 4 = (x + 2)\left(x^2 - 4x + 2\right)$$

$x^2 - 4x + 2$ cannot be factorised any further and we are left with

$(x + 2)\left(x^2 - 4x + 2\right) = 0$

WORKED EXAMPLE 14: SOLVING CUBIC EQUATIONS CONTINUED

Step 3: Solve the equation

$$(x+2)\left(x^2-4x+2\right)=0$$
$$(x+2)=0 \text{ or } \left(x^2-4x+2\right)=0$$

Step 4: Apply the quadratic formula for the second bracket

Always write down the formula first and then substitute the values of a,b and c.

$$a=1; \qquad b=-4; \qquad c=2$$

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2(1)} \\ &= \frac{4\pm\sqrt{8}}{2} \\ &= 2\pm\sqrt{2} \end{aligned}$$

Step 5: Final solutions

$x=-2$ or $x=2\pm\sqrt{2}$

6 SUMMARY

Terminology	
Expression	A term or group of terms consisting of numbers, variables and the basic operators $(+, -, \times, \div)$.
Univariate expression	An expression containing only one variable.
Root/Zero	A root, also referred to as the "zero", of an equation is the value of x such that $f(x) = 0$ is satisfied.
Polynomial	An expression that involves one or more variables having different powers and coefficients. $a_nx^n + \ldots + a_2x^2 + a_1x + a_0,$ where $n \in \mathbb{N}_0$
Monomial	A polynomial with one term. For example, $7a^2b$ or $15xyz^2$
Binomial	A polynomial that has two terms. For exmaple $2x + 5z$ or $26 - g^2k$
Trinomial	A polynomial that has three terms. For example, $a - b + c$ or $4x^2 + 17xy - y^3$
Degree/Order	The degree, also called the order, of a univariate polynomial is the value of the highest exponent in the polynomial. For example, $7p - 12p^2 + 3p^5 + 8$ has degree of 5.

- Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Remainder theorem: a polynomial $p(x)$ divided by $cx-d$ gives a remainder of $p(\frac{d}{c})$.
- Factor theorem: if the polynomial $p(x)$ is divided by $cx-d$ and the remainder, $p(\frac{d}{c})$, is equal to zero, then $cx-d$ is a factor of $p(x)$.
- Converse of the factor theorem: if $cx-d$ is a factor of $p(x)$, then $p(\frac{d}{c}) = 0$.

- Synthetic division:

$$\begin{array}{c|cccc} & q_2 & q_1 & q_0 & R \\ \hline \frac{d}{c} & a_3 & a_2 & a_1 & a_0 \end{array}$$

We determine the coefficients of the quotient by calculating:

$$\begin{aligned} q_2 &= a_3 + \left(q_3 \times \frac{d}{c}\right) \\ &= a_3 \quad \text{(since } q_3 = 0) \\ q_1 &= a_2 + \left(q_2 \times \frac{d}{c}\right) \\ q_0 &= a_1 + \left(q_1 \times \frac{d}{c}\right) \\ R &= a_0 + \left(q_0 \times \frac{d}{c}\right) \end{aligned}$$

7 EXERCISES

7.1 Exercise 1

1. Given $f(x) = 2x^3 + 3x^2 - 1$. Determine whether the following statements are true or false. If false, provide the correct statement.

    1.1 $f(x)$ is a trinomial.

    1.2 The coeffiecient of the x is 0.

    1.3 $f(\frac{1}{2}) = \frac{1}{12}$

    1.4 $f(x)$ is of degree 3.

    1.5 The constant term is 1.

    1.6 $f(x)$ will have 3 real roots.

2. Given $g(x) = 2x^3 - 9x^2 + 7x + 6$, determine the following:

    2.1 The number of terms in $g(x)$

    2.2 The degree of $g(x)$

    2.3 The coefficient of the x^2 term

    2.4 The constant term.

3. Determine which of the following expressions are polynomials and which are not. For those that are not polynomials, give reasons.

    3.1 $y^3 + \sqrt{5}$

    3.2 $-x^2 - x - 1$

    3.3 $4\sqrt{k} - 9$

    3.4 $\frac{2}{p} + p + 3$

    3.5 $x(x-1)(x-2) - 2$

    3.6 $(\sqrt{m} - 1)(\sqrt{m} + 1)$

    3.7 $t^0 - 1$

    3.8 $16y^7$

    3.9 $-\frac{x^3}{2} + 5x^2 + \frac{x}{3} - 11$

    3.10 $4b^0 + 3b^{-1} + 5b^2 - b^3$

4. Solve the following quadratic equations by factorisation. Answers may be left in surd form, where applicable.

   4.1 $7p^2 + 14p = 0$

   4.2 $k^2 + 5k - 36 = 0$

   4.3 $400 = 16h^2$

   4.4 $(x - 1)(x + 10) + 24 = 0$

   4.5 $y^2 - 5ky + 4k^2$

5. Solve the following equations by completing the square:

   5.1 $p^2 + 10p - 2 = 0$

   5.2 $2(6y + y^2) = -4$

   5.3 $x^2 + 5x + 9 = 0$

   5.4 $f^2 + 30 = 2(10 - 8f)$

   5.5 $3x^2 + 6x - 2 = 0$

6. Solve the following using the quadratic formula:

   6.1 $3m^2 + m - 4 = 0$

   6.2 $2t^2 + 6t + 5 = 0$

   6.3 $y^2 - 4y + 2 = 0$

   6.4 $3f - 2 = -2f^2$

7. Factorise the following:

   7.1 $27p^3 - 1$

   7.2 $16 + \frac{2}{x^3}$

7.2 Exercise 2

1. Factorise the following:

   1.1 $p^3 - 1$

   1.2 $t^3 + 27$

   1.3 $64 - m^3$

   1.4 $k - 125k^4$

   1.5 $8a^6 - b^9$

   1.6 $8 - (p + q)^3$

2. For each of the following:
   1) Use long division to determine the quotient $Q(x)$ and the remainder $R(x)$.
   2) Write $a(x)$ in the form $a(x) = b(x) \cdot Q(x) + R(x)$.
   3) Check your answer by expanding the brackets to get back to the original cubic polynomial.

   2.1 $a(x) = x^3 + 2x^2 + 3x + 7$ is divided by $(x + 1)$

   2.2 $a(x) = 1 + 4x^2 - 5x - x^3$ and $b(x) = x + 2$

   2.3 $a(x) = 2x^3 + 3x^2 + x - 6$ and $b(x) = x - 1$

   2.4 $a(x) = x^3 + 2x^2 + 5$ and $b(x) = x - 1$

   2.5 $x - 1$ is divided into $a(x) = x^4 + 2x^3 - 3x^2 + 5x + 4$

   2.6 $\frac{a(x)}{b(x)} = \frac{5x^4+3x^3+6x^2+x+2}{x^2-2}$

   2.7 $a(x) = 3x^3 - x^2 + 2x + 1$ is divided by $(3x - 1)$

   2.8 $a(x) = 2x^5 + x^3 + 3x^2 - 4$ and $b(x) = x + 2$

3. Use synthetic division to determine the quotient $Q(x)$ and the remainder $R(x)$ when $f(x)$ is divided by $g(x)$:

   3.1 $f(x) = x^2 + 5x + 1$; $g(x) = x + 2$

   3.2 $f(x) = x^2 - 5x - 7$; $g(x) = x - 1$

   3.3 $f(x) = 2x^3 + 5x - 4$; $g(x) = x - 1$

   3.4 $f(x) = 19 + x^2 + 8x$; $g(x) = x + 3$

   3.5 $f(x) = x^3 + 2x^2 + x - 10$; $g(x) = x - 1$

   3.6 $f(x) = 4x^3 + 4x^2 - x - 2$; $g(x) = 2x - 1 = 2\left(x - \frac{1}{2}\right)$

   3.7 $f(x) = 5x + 22 + 2x^3 + x^2$; $g(x) = 2x + 3 = 2\left(x + \frac{3}{2}\right)$

   3.8 $f(x) = 2x^3 + 7x^2 + 2x - 3$; $g(x) = x + 3$

7.3 Exercise 3

1. Use the remainder theorem to determine the remainder R when $g(x)$ is divided by $h(x)$:

   1.1 $g(x) = x^3 + 4x^2 + 11x - 5$; $h(x) = x - 1$

   1.2 $g(x) = 2x^3 - 5x^2 + 8$; $h(x) = 2x - 1$

   1.3 $g(x) = 4x^3 + 5x^2 + 6x - 1$; $h(x) = x + 2$

   1.4 $g(x) = -5x^3 - x^2 - 10x + 9$; $h(x) = 5x + 1$

   1.5 $g(x) = x^4 + 5x^2 + 2x - 8$; $h(x) = x + 1$

   1.6 $g(x) = 3x^5 - 8x^4 + x^2 + 2$; $h(x) = 2 - x$

1.7 $g(x) = 2x^{100} - x - 1$; $h(x) = x + 1$

2. Determine the value of t if $x^3 + tx^2 + 8x + 21$ divided by $x + 1$ gives a remainder of 16.
3. Determine the value of m if $2x^3 - 7x^2 + mx - 26$ divided by $x - 2$ gives a remainder of -24.
4. If $x^5 - 2x^3 - kx - 1$ is divided by $x - 1$ and the remainder is $-\frac{1}{2}$, find the value of k.
5. Determine the value of p if $18x^3 + px^2 - 8x + 9$ divided by $2x - 1$ gives a remainder of 6.
6. If $x^3 + x^2 - x + b$ is divided by $x - 2$ and the remainder is $2\frac{1}{2}$, calculate the value of b.
7. Calculate the value of h if $3x^5 + hx^4 + 10x^2 - 21x + 12$ divided by $x - 2$ gives a remainder of 10.
8. If $x^3 + 8x^2 + mx - 5$ is divided by $x + 1$ and the remainder is n, express m in terms of n.
9. When the polynomial $2x^3 + px^2 + qx + 1$ is divided by $x + 1$ or $x - 4$, the remainder is 5. Determine the values of p and q

7.4 Exercise 4

1. Find the remainder when $4x^3 - 4x^2 + x - 5$ is divided by $x + 1$
2. Use the factor theorem to factorise $x^3 - 3x^2 + 4$ completely.
3. $f(x) = 2x^3 + x^2 - 5x + 2$

   3.1 Find $f(1)$

   3.2 Factorise $f(x)$ completely
4. Use the factor theorem to determine all the factors of the following expression: $x^3 + x^2 - 17x + 15$
5. Complete: If $f(x)$ is a polynomial and p is a number such that $f(p) = 0$, then $(x - p)$ is...
6. Given $f(1) = f(3) = f(-1) = 0$:

   6.1 Write $f(x)$ in the form $f(x) = ax^3 + bx^2 + cx + d$

   6.2 Suppose $f(x) = x^3 + mx^2 - x + 3$ and $f(-2) = -15$, determine the value of m
7. Given: $f(x) = 2x^3 + ax^2 - 21x - 36$ is divisible by $(x + 4)$, determine the value of a.
8. Solve for a and b in each of the following:

   8.1 $(x - 2)$ and $(x + 1)$ are factors of $x^3 + ax^2 + bx + 8$

   8.2 $(x - 1)$ and $(x + 1)$ are factors of $ax^3 + bx^2 + 5x - 5$
9. Given: $f(x) = 4x^3 - 7x^2 - 7x + 30$ and $g(x) = 4x + 5$.

   9.1 What is the remainder when $f(x)$ is divided by $g(x)$?

   9.2 If $f(x) = 4x^3 - 7x^2 - 7x + k$, find k so that $g(x)$ becomes a factor of $f(x)$

7.5 Exercise 5

Solve the following cubic equations:

1. $x^3 + x^2 - 16x = 16$
2. $-n^3 - n^2 + 22n + 40 = 0$
3. $y(y^2 + 2y) = 19y + 20$
4. $k^3 + 9k^2 + 26k + 24 = 0$
5. $x^3 + 2x^2 - 50 = 25x$
6. $-p^3 + 19p = 30$
7. $6x^2 - x^3 = 5x + 12$

8 ANSWERS FOR EXERCISES

8.1 Exercise 1

1. 1.1 True

   1.2 True

   1.3 False, $f(\frac{1}{2}) = 0$

   1.4 True

   1.5 False, -1

   1.6 True

2. 2.1 4

   2.2 3

   2.3 -9

   2.4 6

3. 3.1 Cubic polynomial

   3.2 Quadratic polynomial

   3.3 Not a polynomial; in $k^{\frac{1}{2}}$ the exponent is not a natural number.

   3.4 Not a polynomial; in p^{-1} the exponent is not a natural number.

   3.5 Cubic polynomial

   3.6 Linear polynomial

   3.7 Zero Polynomial

   3.8 Polynomial; degree 7

   3.9 Cubic polynomial

   3.10 Not a polynomial; in b^{-1} the exponent is not a natural number.

4. 4.1 $p = 0$ or $p = -2$

   4.2 $k = 4$ or $k = -9$

   4.3 $h = \pm 5$

   4.4 $x = -7$ or $x = -2$

   4.5 $y = 4k$ or $y = k$

5. 5.1 $p = -5 - 3\sqrt{3}$ or $p = -5 + 3\sqrt{3}$

   5.2 $y = -3 \pm \sqrt{7}$

5.3 No real solution

5.4 $f = -8 \pm 3\sqrt{6}$

5.5 $x = -1 \pm \sqrt{\frac{5}{3}}$

6. 6.1 $m = 1$ or $m = -\frac{4}{3}$

   6.2 No real solution

   6.3 $y = 2 + \sqrt{2}$ or $y = 2 - \sqrt{2}$

   6.4 $f = -2$ or $f = \frac{1}{2}$

7. 7.1 $(3p - 1)(9p^2 + 3p + 1)$

   7.2 $2(2 + \frac{1}{x})(4 - \frac{2}{x} + \frac{1}{x^2})$

8.2 Exercise 2

1. 1.1 $(p - 1)(p^2 + p + 1)$

   1.2 $(t + 3)(t^2 - 3t + 9)$

   1.3 $(4 - m)(16 + 4m + m^2)$

   1.4 $k(1 - 5k)(1 + 5k + 25k^2)$

   1.5 $(2a^2 - b^3)(4a^4 + 2a^2b^3 + b^6)$

   1.6 $(2 - p - q)(4 + 2p + 2q + p^2 + 2pq + q^2)$

2. 2.1 $Q(x) = x^2 + x + 2$
   $R(x) = 5$
   $a(x) = (x + 1)(x^2 + x + 2) + 5$

   2.2 $Q(x) = -x^2 + 6x - 17$
   $R(x) = 35$
   $a(x) = (x + 2)(-x^2 + 6x - 17) + 35$

   2.3 $Q(x) = 2x^2 + 5x + 6$
   $R(x) = 0$
   $a(x) = (x - 1)(2x^2 + 5x + 6)$

   2.4 $Q(x) = x^2 + 3x + 3$
   $R(x) = 8$
   $a(x) = (x - 1)(2x^2 + 3x + 3) + 8$

   2.5 $Q(x) = x^3 + 3x^2 + 5$
   $R(x) = 9$
   $a(x) = (x - 1)(x^3 + 3x^2 + 5) + 9$

2.6 $Q(x) = 5x^2 + 3x + 16$

$R(x) = 7x + 34$

$a(x) = (x^2 - 2)(5x^2 + 3x + 16) + 7x + 34$

2.7 $Q(x) = x^2 + \frac{2}{3}$

$R(x) = \frac{5}{3}$

$a(x) = (3x - 1)\left(x^2 + \frac{2}{3}\right) + \frac{5}{3}$

2.8 $Q(x) = 2x^4 - 4x^3 + 9x^2 - 15x + 30$

$R(x) = -64$

$a(x) = (x + 2)(2x^4 - 4x^3 + 9x^2 - 15x + 30) - 64$

3. 3.1 $Q(x) = x + 3$

$R(x) = -5$

3.2 $Q(x) = x - 4$

$R(x) = -11$

3.3 $Q(x) = 2x^2 + 2x + 7$

$R(x) = 3$

3.4 $Q(x) = x + 5$

$R(x) = 4$

3.5 $Q(x) = x^2 + 3x + 4$

$R(x) = -6$

3.6 $Q(x) = 4x^2 + 6x + 2$

$R(x) = -1$

3.7 $Q(x) = 2x^2 - 2x + 8$

$R(x) = 10$

3.8 $Q(x) = 2x^2 + x - 1$

$R(x) = 0$

8.3 Exercise 3

1. 1.1 $R = 11$

1.2 $R = 7$

1.3 $R = -25$

1.4 $R = 11$

1.5 $R = -4$

1.6 $R = -26$

1.7 $R = 2$

2. $t = 4$
3. $m = 7$
4. $k = -\frac{3}{2}$
5. $p = -5$
6. $b = -\frac{15}{2}$
7. $h = -6$
8. $m = 2 - n$
9. $p = -5$
   $q = -11$

8.4 Exercise 4

1. -14
2. $(x+1)(x-2)^2$
3. 3.1 0
   3.2 $(x-1)(2x-1)(x+2)$
4. $(x-1)(x+5)(x-3)$
5. A factor of $f(x)$
6. 6.1 $f(x) = x^3 - 3x^2 - x + 3$
   6.2 $m = -3$
7. $a = 5$
8. 8.1 $a = -5$ and $b = 2$
   8.2 $a = -5$ and $b = 5$
9. 9.1 20
   9.2 $k = 10$

8.5 Exercise 5

1. $x = -4$ or $x = -1$ or $x = 4$
2. $n = -4$ or $n = -2$ or $n = 5$
3. $y = -5$ or $y = -1$ or $y = 4$
4. $k = -4$ or $k = -3$ or $k = -2$
5. $x = -5$ or $x = -2$ or $x = 5$
6. $p = -5$ or $p = 2$ or $p = 3$
7. $x = -1$ or $x = 3$ or $x = 4$