CHAPTER 13

Pythagoras' Theorem

CONTENTS

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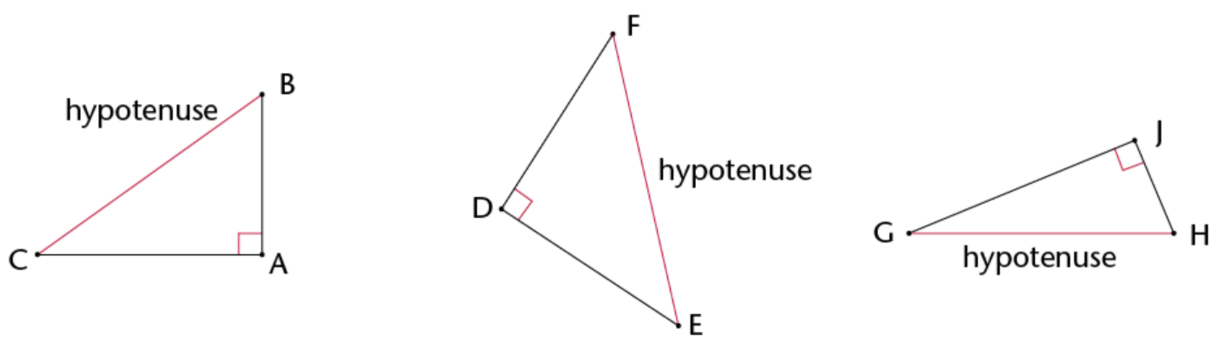
MathU Teaching® Emporium

1 PYTHAGORAS' THEOREM

theorem is a rule or a statement that has been proved through reasoning. Pythagoras' Theorem is a rule that applies only to **right-angled triangles**. The theorem is named after the Greek mathematician, Pythagoras.**hypotenuse**.

Pythagoras (569-475 BC)

Pythagoras was an influential mathematician. like many Greek mathematicians of 2500 years ago, he was also a philosopher and a scientist. He formulated the best known theorem, today known as Pythagoras' Theorem. However, the theorem had already been in use 1000 years earlier, by the Chinese and Babylonians.

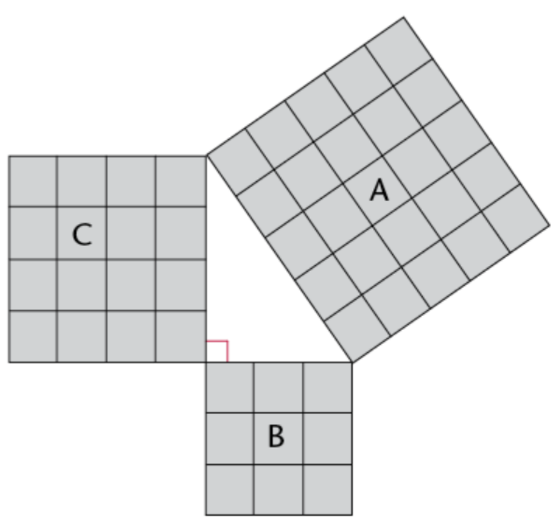


The **hypotenuse** is the side opposite the 90° angle in a right-angled triangle. It is always the longest side.

How to say it:

'high - pot - eh - news'

- The figure shows a right-angled triangle with squares on each of the sides.



1. Write down the areas of the following:
    1.1 Square A:
    1.2 Square B:
    1.3 Square C:
2. Add Area of square B + Area of square C:
3. What do you notice about the areas?

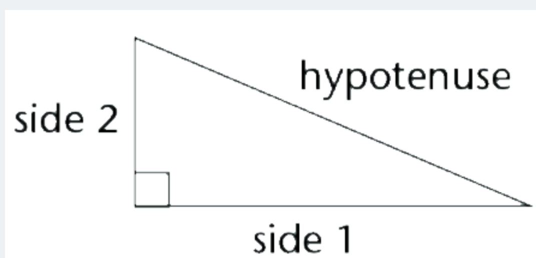
In the previous activity, you should have discovered Pythagoras' Theorem for right-angled triangles.

Note

Pythagoras' Theorem says:

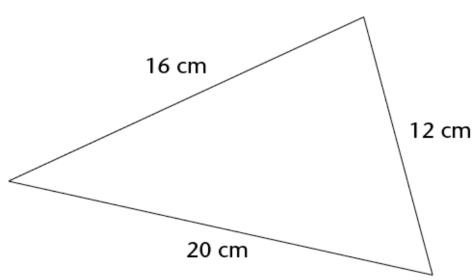
In a right-angled triangle, a square formed on the hypotenuse will have the same area as the sum of the area of the two squares formed on the other sides of the triangle. Therefore:

$(Hypotenuse)^2 = (Side\ 1)^2 + (Side\ 2)^2$

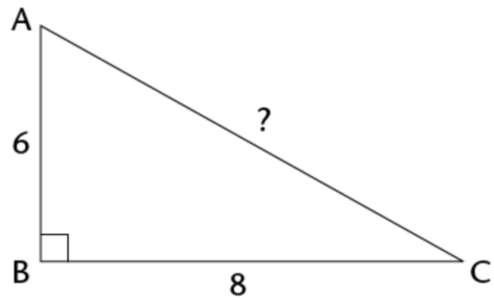


- If a triangle is right-angled, the sides will have the following relationship: $(Hypotenuse)^2 = (Side\ 1)^2 + (Side\ 2)^2$
- If the sides have the relationship: $(Longest\ side)^2 = (Side\ 1)^2 + (Side\ 2)^2$, then the triangle is a right-angled triangle.

Example:



Example:



surd.

Surd form

You pronounce surd so that it rhymes with word.

$\sqrt{5}$ is an example of a number in surd form.

$\sqrt{9}$ is not a surd because you can simplify it:

$\sqrt{9} = 3$

Sets of **whole numbers** that can be used as the sides of a right-angled triangle are known as **Pythagorean triples**, for example:

3-4-5 ;5-12-13 ;7-24-25 ;16-30-34 ;20-21-29

You extend these triples by finding multiples of them. For examples, triples from the 3-4-5 set include the following:

3-4-5 ;6-8-10 ;9-12-15 ;12-16-20

There are many old writings that record Pythagorean triples. For example, from 1900 to 1600 BC the Babylonians had already calculated very large Pythagorean triples, such as:

1 679-2 400-2 929.

How many Pythagorean triples can you find? What is the largest one you can find that is not a multiple of another one?

Pythagoras' Theorem works only for right-angled triangles. But we can also use it to find out whether other triangles are acute or obtuse, as follows.

- **If the square of the longest side is less than the sum of the squares of the two shorter sides, the biggest angle is acute.**

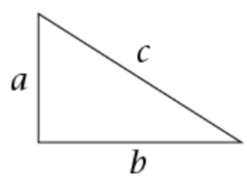
For example, in a 6-8-9 triangle: 6^2 + 8^2 = 100 and 9^2 = 81.

81 is less than 100 therefore the 6-8-9 triangle is acute.

- **If the square of the longest side is more than the sum of the squares of the two shorter sides, the biggest angle is obtuse.**

For example, in a 6-8-11 triangle: 6^2 + 8^2 = 100 and 11^2 = 121.

121 is more than 100 therefore the 6-8-11 triangle is obtuse.



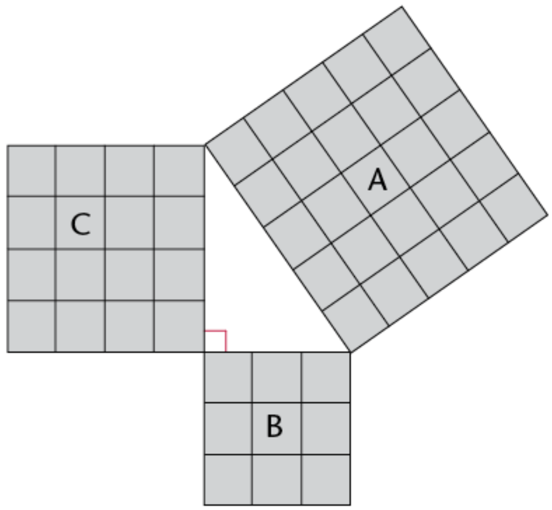
Complete the following table. It is based on the triangle on the right. Decide whether each triangle described is right-angled, acute or obtuse.

a	b	c	$a^2 + b^2$	c^2	Fill ini =, > or <	Type of triangle
3	5	6	$3^2 + 5^2 = 9 + 25 = 34$	6^2 = 36	$a^2 + b^2 > c^2$	Acute
2	4	6			$a^2 + b^2$ .... c^2	
5	7	9			$a^2 + b^2$ .... c^2	
12	5	13			$a^2 + b^2$ .... c^2	
12	16	20	$12^2 + 16^2 = 144 + 256 = 400$	$20^2 = 400$	$a^2 + b^2$ .... c^2	Right-angled
7	9	11			$a^2 + b^2$ .... c^2	
8	12	13			$a^2 + b^2$ .... c^2	

2 EXERCISES

2.1 Exercise 1

1. The figure shows a right-angled triangle with squares on each of the sides.



   (a) Write down the areas of the following:
   Square A:

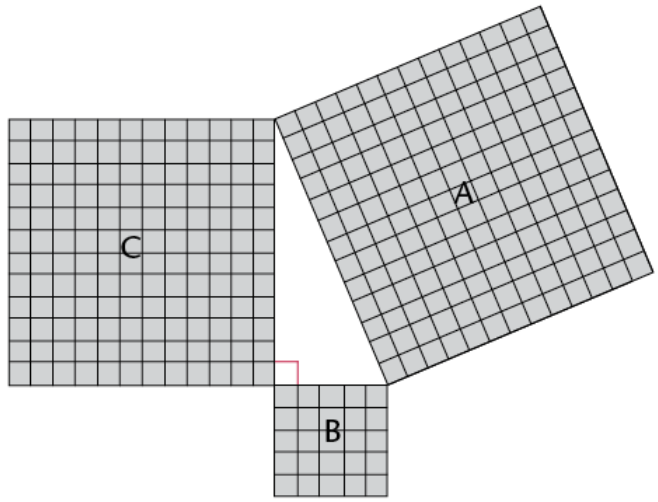
   Square B:

   Square C:

   (b) Add Area of square B + Area of square C:

   (c) What do you notice about the areas?

2. The figure below is similar to the one in question 1. The lengths of the sides of the right-angled triangle are 5 cm and 12 cm.



(a) What is the length of the hypotenuse? Count the squares.

(b) Use the squares to find the following:

Area of A:

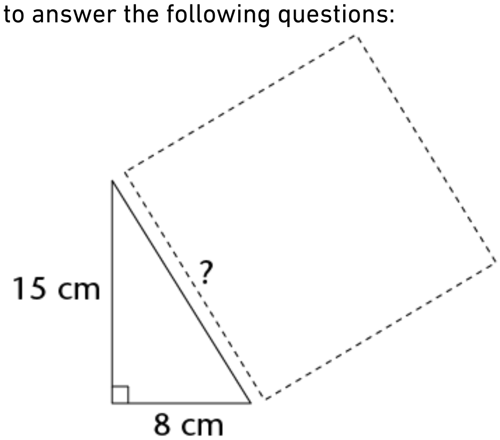
Area of B:

Area of C:

Area of B + Area of C:

(c) What do you notice about the areas? Is it similar to your answer in 1(c)?

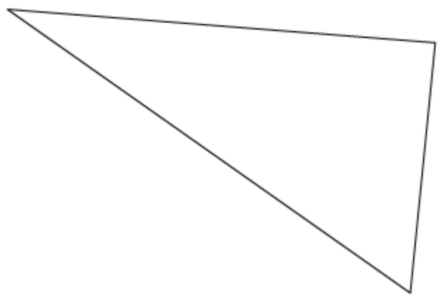
3. A right-angled triangle has side lengths of 8 cm and 15 cm. Use your findings in the previous questions to answer the following questions:



(a) What is the area of the square drawn along the hypotenuse?

(b) What is the length of the triangle's hypotenuse?

2.2 Exercise 2

1. 

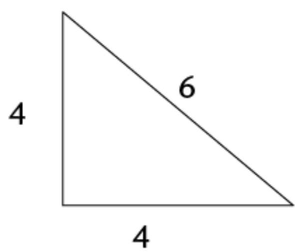
   This triangle's side lengths are 29 mm, 20 mm and 21 mm.

   (a) Prove that it is a right-angled triangle.

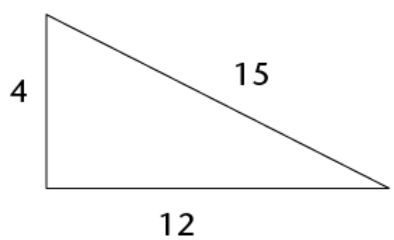
   (b) Copy the triangle and mark the right angle in the diagram

2. Use Pythagoras' Theorem to determine whether these triangles are right-angled. All values are in the same units.

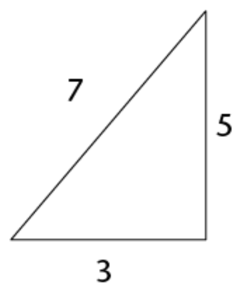
   (a)

   (b)

(c)



3. Determine whether the following side lengths would form right-angled triangles. All values are in the same units.

   (a) $7, 9$ and 12

   (b) $7, 12$ and 14

   (c) $16, 8$ and 10

   (d) $6, 8$ and 10

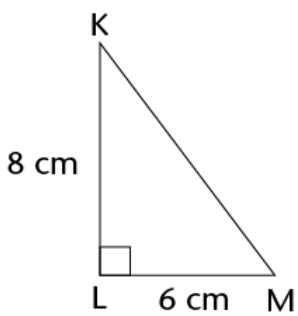
   (e) $8, 15$ and 17

   (f) $16, 21$ and 25

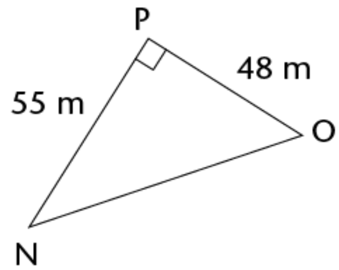
2.3 Exercise 3

1. Find the length of the hypotenuse in each of the triangles below. Leave the answers in surd form where applicable.

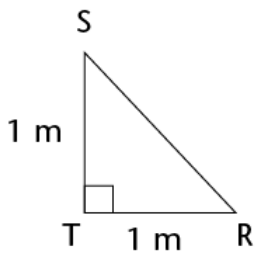
   (a)



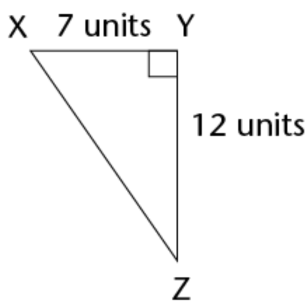
(b)



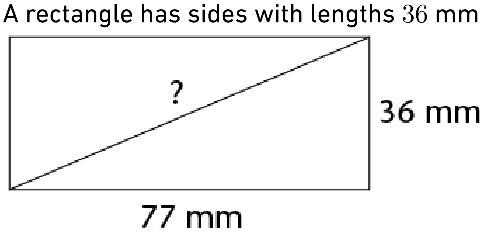
(c)



(d)

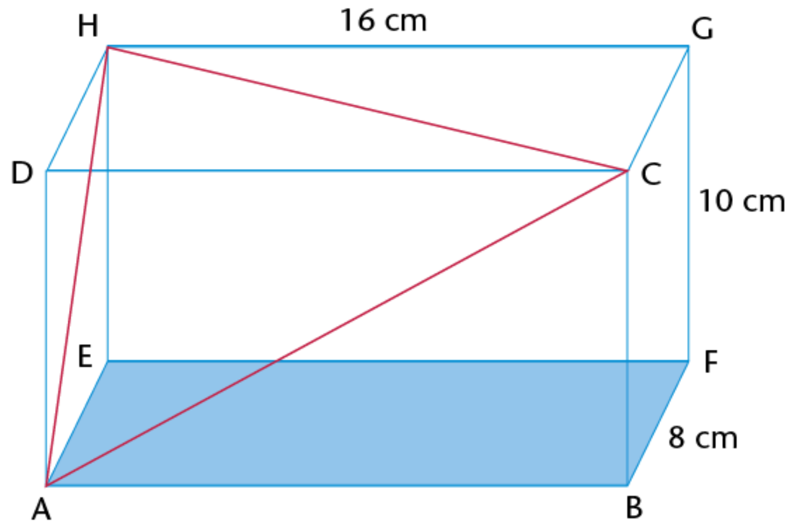


2. A rectangle has sides with lengths 36 mm and 77 mm. Find the length of the rectangle's diagonal.



3. △ABC has $\hat{A} = 90°$, AB = 3 cm and AC = 5 cm. Make a rough sketch of the triangle, and then calculate the length of BC.

4. A rectangular prism is made of glass. It has a length of 16 cm, a height of 10 cm and a breadth of 8 cm. ABCD and EFGH are two of its faces. △ACH has been drawn inside the prism. Is △ACH right-angled? Answer the questions to find out.



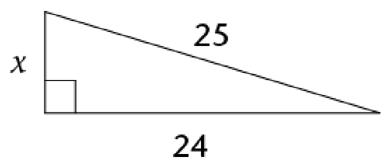
(a) Calculate the length of the sides of △ACH. Note that all three sides of the triangles are diagonals of rectangles. AC is in rectangle ABCD, AH is in ADHE and HC is in HDCG.

(b) Is △ACH right-angled? Explain your answer.

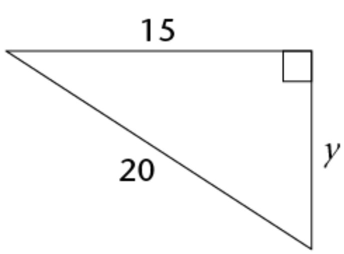
2.4 Exercise 4

1. In the right-angled triangles below, calculate the length of the sides that have not been given. Leave your answers in surd form where applicable.

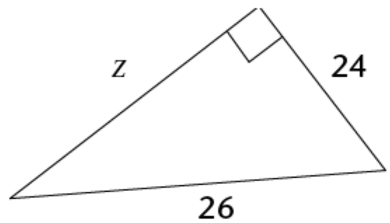
   (a)



   (b)



(c)



2. Calculate the length of the third side of each of the following right-angled triangles. First draw a rough sketch of each of the triangles before you do any calculations. Round off your answers to two decimal places.

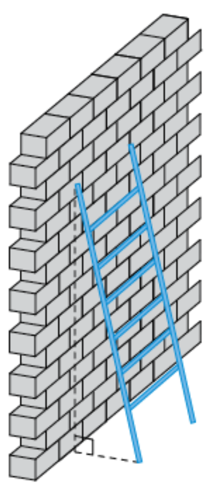
   (a) $\triangle$ ABC has AB = 12 cm, BC = 18 cm and $\hat{A} = 90°$. Calculate AC.

   $\triangle$DEF has $\hat{F} = 90°$, DE = 58 cm and DF = 41 cm. Calculate EF.

   $\triangle$JKL has $\hat{K} = 90°$, JK = 119 m, KL = 167 m. Calculate JL.

   $\triangle$PQR has PQ = 2 cm, QR = 8 cm and $\hat{Q} = 90°$. Calculate PR.

3.

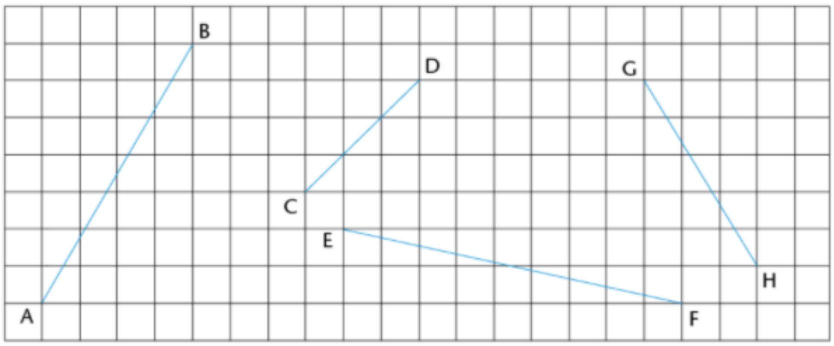


   (a) A ladder of length 5 m is placed at an angle against a wall. The bottom of the ladder is 1 m away from the wall. How far up the wall will the ladder reach? Round off to two decimal places.

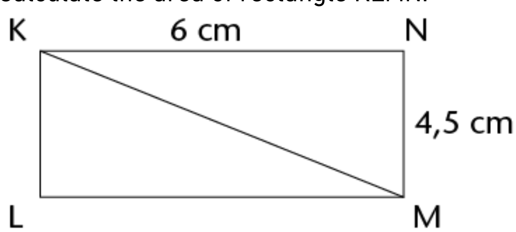
   (b) If the ladder reaches a height of $4,5$ m against the wall, how far away from the wall was it placed? Round off to two decimal places.

2.5 Exercise 5

1. Four lines have been drawn on the grid below. Each square is 1 unit long. Calculate the lengths of the lines: AB, CD, EF and GH. Do the calculations in your exercise book and write the answers below. Leave your answers in surd form.

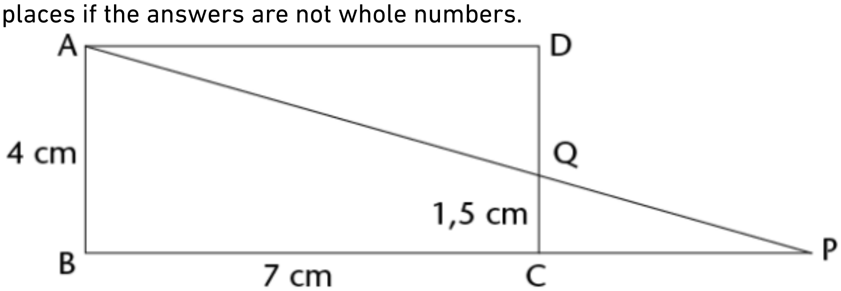


2. (a) Calculate the area of rectangle KLMN.



   (b) Calculate the perimeter of $\triangle$KLM.

3. ABCD is a rectangle with AB = 4 cm, BC = 7 cm and CQ = $1,5$ cm. Round off your answers to two decimal places if the answers are not whole numbers.

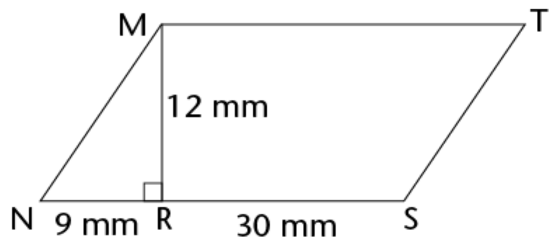


   (a) What is the length of QD?

   (b) If CP = $4,2$ cm, calculate the length of PQ.

   (c) Calculate the length of AQ and the area of $\triangle$AQD.

4. MNST is a parallelogram. NR = 9 mm and MR = 12 mm.



(a) Calculate the area of $\triangle$MNR.

(b) Calculate the perimeter of MNST.

2.6 Exercise 6

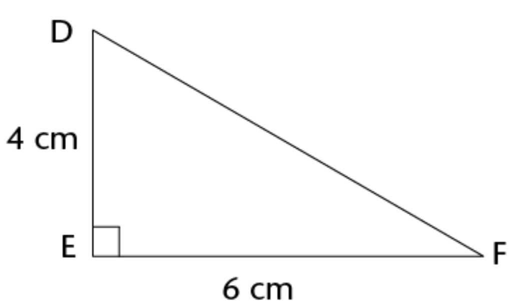
Complete the following table. It is based on the triangle on the right. Decide whether each triangle described is right-angled, acute or obtuse.

a	b	c	a^2+b^2	c^2	Fill in $=, >$ or $<$	Type of triangle
3	5	6	$3^2+5^2=9+25=34$	$6^2=36$	$a^2+b^2<c^2$	Acute
2	4	6			$a^2+b^2....c^2$	
5	7	9			$a^2+b^2....c^2$	
12	5	13			$a^2+b^2....c^2$	
12	16	20	$12^2+16^2=144+256=400$	$20^2=400$	$a^2+b^2=c^2$	Right-angled
7	9	11			$a^2+b^2....c^2$	
8	12	13			$a^2+b^2....c^2$	

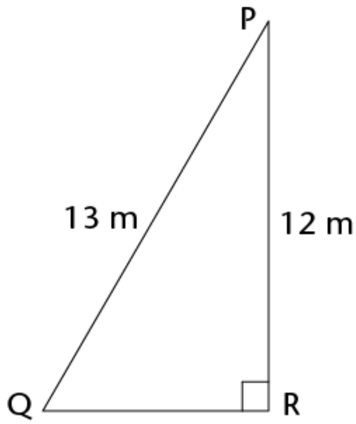
2.7 Exercise 7

1. Write down Pythagoras' Theorem in the way that you best understand it.
2. Calculate the lengths of the missing sides in the following triangles. Leave the answers in surd form if necessary.

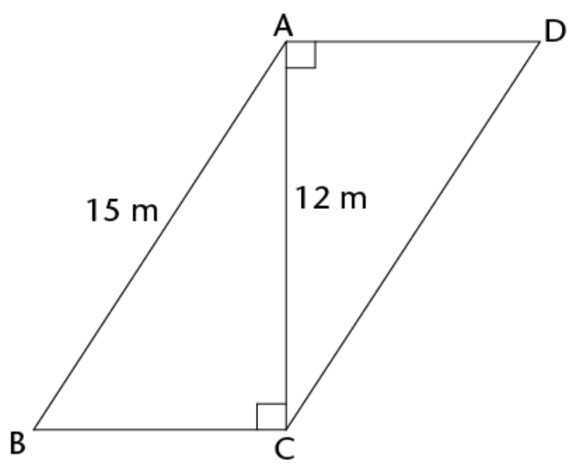
(a)



(b)



3. ABCD is a parallelogram.



(a) Calculate the perimeter of ABCD.

(b) Calculate the area of ABCD.

3 ANSWERS TO EXERCISES

3.1 Exercise 1

1. (a) Square A: $5 \times 5 = 25$ square units
   Square B: $3 \times 3 = 9$ square units
   Square C: $4 \times 4 = 16$ square units

   (b) $9 + 16 = 25$ square units

(c) Area of square B + Area of square C = Area of square A

2. (a) 13 cm

(b) Area of A: $13 \times 13 = 169$ cm^2
Area of B: $5 \times 5 = 25$ cm^2
Area of B: $12 \times 12 = 144$ cm^2
Area of square B + Area of square C = $25 + 144 = 169$ cm^2

(c) The sum of the areas of the two squares formed on the sides of the right angled triangle is equal to the area of the square formed on the hypotenuse.
Yes, it is similar to the answer in 1(c).

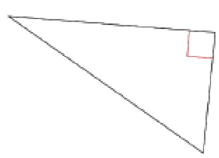
3. (a) 15 cm $\times 15$ cm $= 225$ cm^2
8 cm $\times 8$ cm $= 64$ cm^2
Area of square along hypotenuse = 225 cm^2 $+ 64$ cm^2 $= 289$ cm^2

(b) Length of hypotenuse = $\sqrt{289}$ cm $= 17$ cm

3.2 Exercise 2

1. (a) (Longest side)$^2 = 292^2 = 841$ mm^2
(Side $1)^2$ + (Side $2)^2 = 20^2 + 21^2$
$400 + 441 = 841$ mm^2 $\therefore$ The triangle is right-angled.

(b)



2. (a) $4^2 + 4^2 = 16 + 16 = 32; 6^2 = 36$; not right-angled

(b) $4^2 + 12^2 = 16 + 144 = 160; 15^2 = 225$; not right-angled

(c) $5^2 + 3^2 = 25 + 9 = 34; 7^2 = 49$; not right-angled

3. (a) $7^2 = 49; 9^2 = 81; 12^2 = 144; 49 + 81 = 130$; not right-angled

(b) $7^2 = 49; 12^2 = 144; 14^2 = 196; 49 + 144 = 193$; not right-angled

(c) $16^2 = 256; 8^2 = 64; 10^2 = 100; 100 + 64 = 164$; not right-angled

(d) $6^2 = 36; 8^2 = 64; 10^2 = 100; 36 + 64 = 100$; right-angled

(e) $8^2 = 64; 15^2 = 225; 17^2 = 289; 64 + 225 = 289$; right-angled

(f) $16^2 = 256; 21^2 = 441; 25^2 = 625; 256 + 441 = 697$; not right-angled

3.3 Exercise 3

1. (a) $\text{KM}^2 = 8^2 + 6^2 = 64 + 36 = 100$ cm^2; KM = $\sqrt{100}$ cm = 10 cm

   (b) $\text{NO}^2 = 55^2 + 48^2 = 3\,025 + 2\,304 = 5\,329$ m^2; NO = $\sqrt{5\,329}$ m = 73 m

   (c) $\text{SR}^2 = 1^2 + 1^2 = 1 + 1 = 2$ m^2; SR = $\sqrt{2}$ m

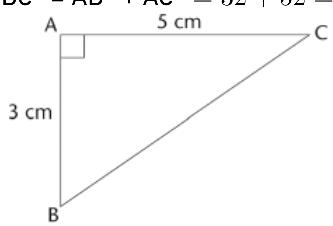
   (d) $\text{XZ}^2 = 7^2 + 12^2 = 49 + 144 = 193$ units^2; XZ = $\sqrt{193}$ units

2. $77^2 + 36^2 = 5\,929 + 1\,296$

   $= 7\,225$ mm^2

   Diagonal $= \sqrt{7\,225}$ mm $= 85$ mm

3. BC^2 = AB^2 + AC^2 $= 32 + 52 = 9 + 25 = 34$ cm^2; BC = $3\sqrt{4}$ cm



4. (a) HC^2 = HG^2 + CG^2 $= 16^2 + 8^2 = 256 + 64 = 320$ cm^2; HC = $\sqrt{320}$ cm
   AH^2 = DH^2 + AD^2 $= 8^2 + 10^2 = 64 + 100 = 164$ cm^2; AH = $\sqrt{164}$ cm

   (b) AC^2 = AB^2 + BC^2 $= 16^2 + 10^2 = 256 + 100 = 356$ cm^2; AC = $\sqrt{356}$ cm

   (c) 320 cm^2 $+ 164$ cm^2 $= 484$ cm^2
   No, $\triangle$ACH is not a right-angled triangle.

3.4 Exercise 4

1. (a) $x^2 = 25^2 – 24^2 = 625 – 576 = 49; x = \sqrt{49} = 7$

   (b) $y^2 = 20^2 – 15^2 = 400 – 225 = 175; y = \sqrt{175}$

   (c) $z^2 = 26^2 – 24^2 = 676 – 576 = 100; z = \sqrt{100} = 10$

2. (a) AC^2 $= 18^2 – 12^2 = 324 – 144 = 180$ cm^2
   $\therefore$ AC = 180 cm $\approx 13,42$ cm

   (b) EF^2 $= 58^2 – 41^2 = 3\,364 – 1\,681 = 1\,683$ cm^2
   $\therefore$ EF = $\sqrt{1683}$ cm $\approx 41,02$ cm

   (c) JL^2 $= 119^2 + 167^2 = 14\,161 + 27\,889 = 42\,050$ m^2
   $\therefore$ JL = $42\,050$ m $\approx 205,06$ m

   (d) PR^2 $= 2^2 + 8^2 = 4 + 64 = 68$ cm^2
   $\therefore$ PR = 68 cm $\approx 8,25$ cm

3. (a) height against wall = h

   $h^2 = 5^2 – 1^2 = 25 – 1 = 24$ m^2

   $h = \sqrt{24}$ m $\approx 4,90$ m

   The ladder reaches $4,9$ m up the wall.

   (b) $5^2 = 4,5^2 + d^2; 25 = 20,25 + d^2; 4,75$ m$^2 = d^2; d = (4,75) \approx 2,18$m

3.5 Exercise 5

1. AB = $\sqrt{65}$ units

   CD = $\sqrt{18}$ units

   EF = $\sqrt{85}$ units

   GH = $\sqrt{34}$ units

2. (a) 6 cm $\times 4,5$ cm = 27 cm^2

   (b) KM$^2 = 6^2 + 4,5^2 = 36 + 20,25 = 56,25$

   KM = $(\sqrt{56,25}) = 7,5$ cm

   Perimeter: $6 + 4,5 + 7,5 = 18$ cm

3. (a) QD = $4 – 1,5 = 2,5$ cm

   (b) PQ$^2 = 4,2^2 + 1,5^2 = 17,64 + 2,25 = 19,89$ cm^2; PQ = $(\sqrt{19,89}) \approx 4,46$ cm

   (c) AQ$^2 = 7^2 + 2,5^2 = 49 + 6,25 = 55,25$ cm^2; AQ = $(\sqrt{55,25})$ cm $\approx 7,43$ cm

   $A = 12\ b \times h = \frac{(7\times 2,5)}{2} = 8,75$ cm^2

4. (a) $12\ b \times h = \frac{9\times 12}{2} = 54$ mm^2

   (b) MN$^2 = 9^2 + 12^2 = 81 + 144 = 225$ mm^2; MN = $\sqrt{225} = 15$ mm

   Perimeter = $2(15 + 9 + 30) = 2 \times 54 = 108$ mm

3.6 Exercise 6

a	b	c	$a^2 + b^2$	c^2	Fill in $=, >$or$<$	Type of triangle
3	5	6	$3^2 + 5^2 = 9 + 25 = 34$	$6^2 = 36$	$a^2 + b^2 < c^2$	Acute
2	4	6	$2^2 + 4^2 = 4 + 16 = 20$	$6^2 = 36$	$a^2 + b^2 < c^2$	Acute
5	7	9	$5^2 + 7^2 = 25 + 49 = 74$	$9^2 = 81$	$a^2 + b^2 < c^2$	Acute
12	5	13	$12^2 + 5^2 = 144 + 25 = 169$	$13^2 = 169$	$a^2 + b^2 = c^2$	Right-angled
12	16	20	$12^2 + 16^2 = 144 + 256 = 400$	$20^2 = 400$	$a^2 + b^2 = c^2$	Right-angled
7	9	11	$7^2 + 9^2 = 49 + 81 = 130$	$11^2 = 121$	$a^2 + b^2 > c^2$	Obtuse
8	12	13	$8^2 + 12^2 = 64 + 144 = 208$	$13^2 = 169$	$a^2 + b^2 > c^2$	Obtuse

3.7 Exercise 7

1. Sample answer: In a right-angled triangle, a square formed on the hypotenuse will have the same area as the sum of the area of the two squares formed on the other sides of the triangle.

2. (a) DF$^2 = 42 + 62 = 16 + 36 = 52$ cm^2
   DF = $\sqrt{52}$ cm = 4×13 cm = $2\sqrt{13}$ cm

   (b) QR$^2 = 13^2 – 12^2 = 169 – 144 = 25$ m^2
   QR = $\sqrt{25}$ cm = 5 cm

3. (a) AD$^2 = 225 – 144 = 81$ m^2
   AD = 9 m Perimeter = $2(9 + 15) = 48$ m

   (b) $A = b \times h$

   $... = 9 \times 12$

   $... = 108$m^2